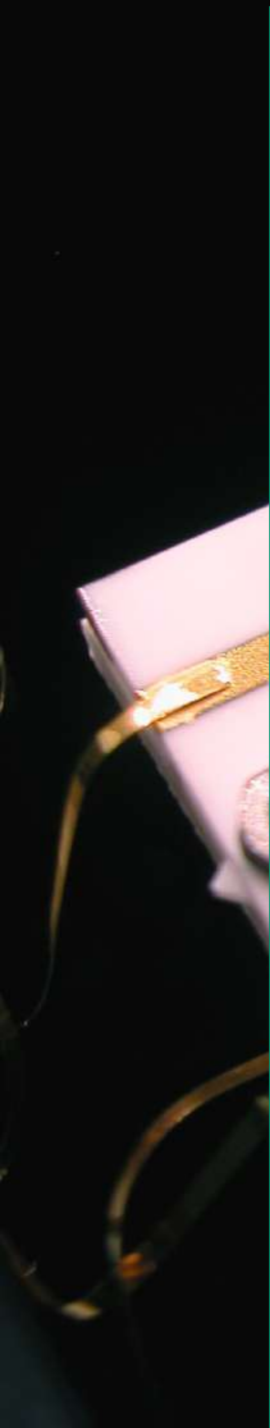
# Summary[#]

* Lecture 1:
  - ◊ ion trap basics
  - ◊ coherent quantum state manipulation and laser cooling
* Lecture 2:
  - ◊ elements of quantum information processing and quantum computers
* Lecture 3:
  - ◊ quantum-limited metrology

# some overlap with Ferdinand Schmidt-Kaler

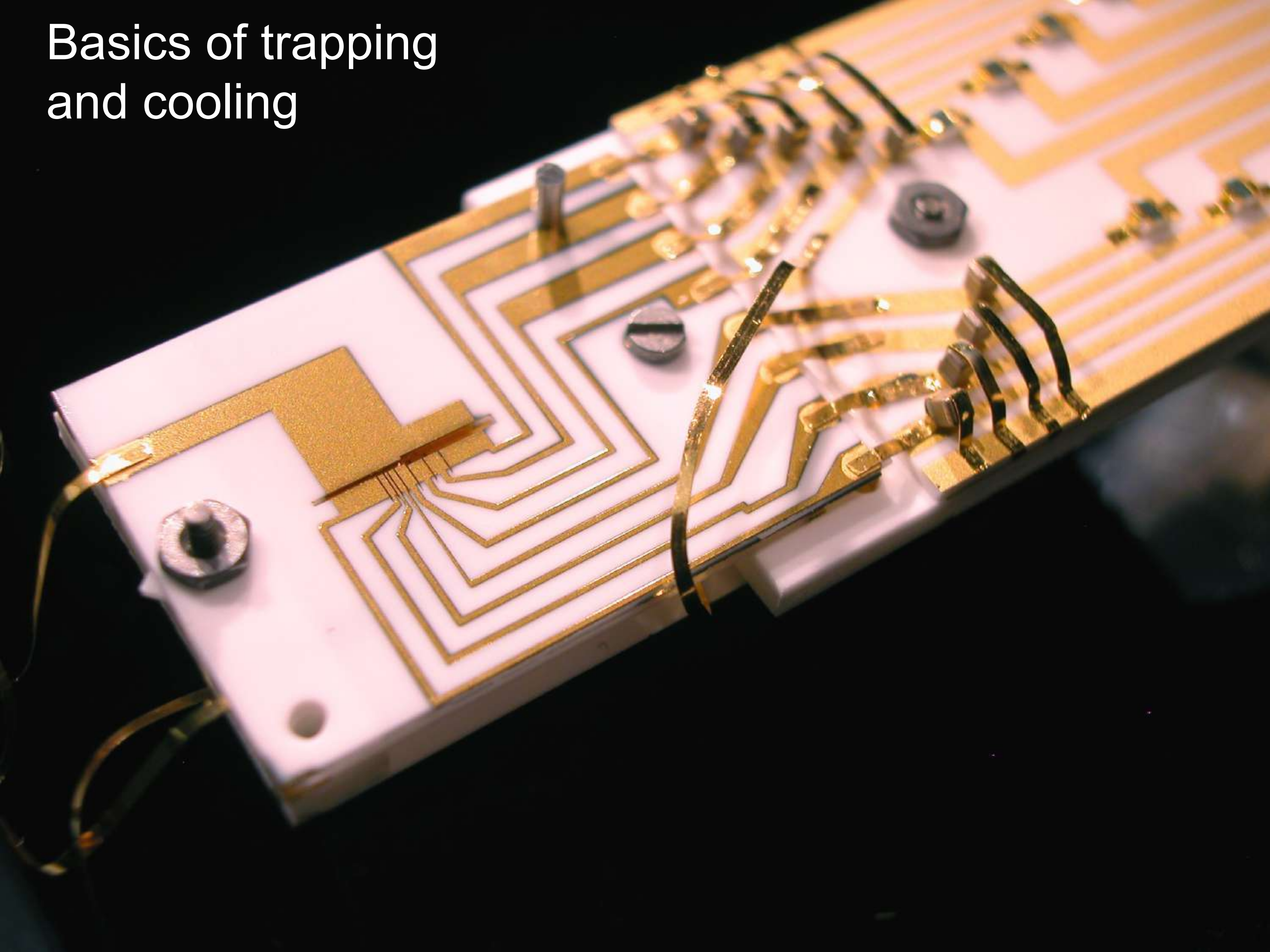
Dave Wineland
University of Oregon, Eugene, Oregon



## Atomic ion experimental groups pursuing quantum information & metrology

Aarhus
AFRL-Rome
Amherst
ARL-Adelphi
Basel
Berkeley
Bonn
Buenos Aires ←
Citadel
Clemson
Denison
Duke
Erlangen
ETH (Zürich)
Freiburg
Georgia Tech
GTRI
Griffith
Hannover
Honeywell
Imperial (London)
Indiana
Innsbruck
IonQ
Lincoln Labs
Marseille
MIT
Munich/Garching
NIST, Boulder
Northwestern
NPL
Osaka
Oxford
Paris (Université Paris)
Pretoria, S. Africa
PTB
Saarbrucken
Sandia National Lab
Siegen
Simon Fraser
Singapore
SK Telecom, S. Korea
Sussex
Sydney
Tsinghua (Bejing)
UCLA
U. Oregon
U. Washington
Waterloo
Weizmann Institute
Williams

# Basics of trapping and cooling



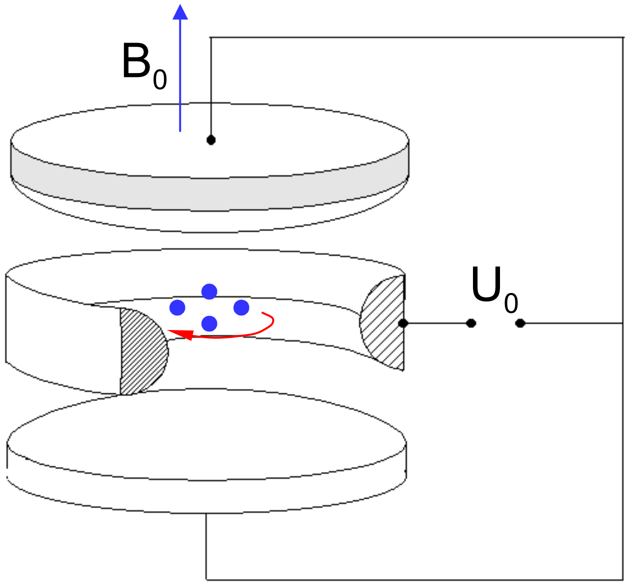
# Ion traps

"Earnshaw's theorem": ~ In a charge free region, cannot confine a charged particle with static electric fields.

Proof: For confinement, must have $(\partial^2(q\Phi)/\partial^2 x_i)_{\text{trap location}} > 0$ ($x_i \in \{x,y,z\}$)

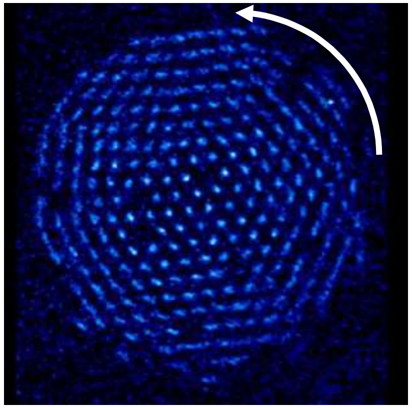
but Laplace's equation: $\nabla^2\Phi = 0$, $\therefore$ cannot (simultaneously) satisfy condition for all $x_i$.

## Solution 1: Penning trap:




$q\Phi \propto U_0\,[2z^2 - x^2 - y^2]$

with magnetic field <u>$B_0$ along z</u>



View along z

For cold ions:
<u>in x-y plane</u>:
small cyclotron orbits
+ overall rotation
<u>normal to plane (z)</u>
electric harmoinic well

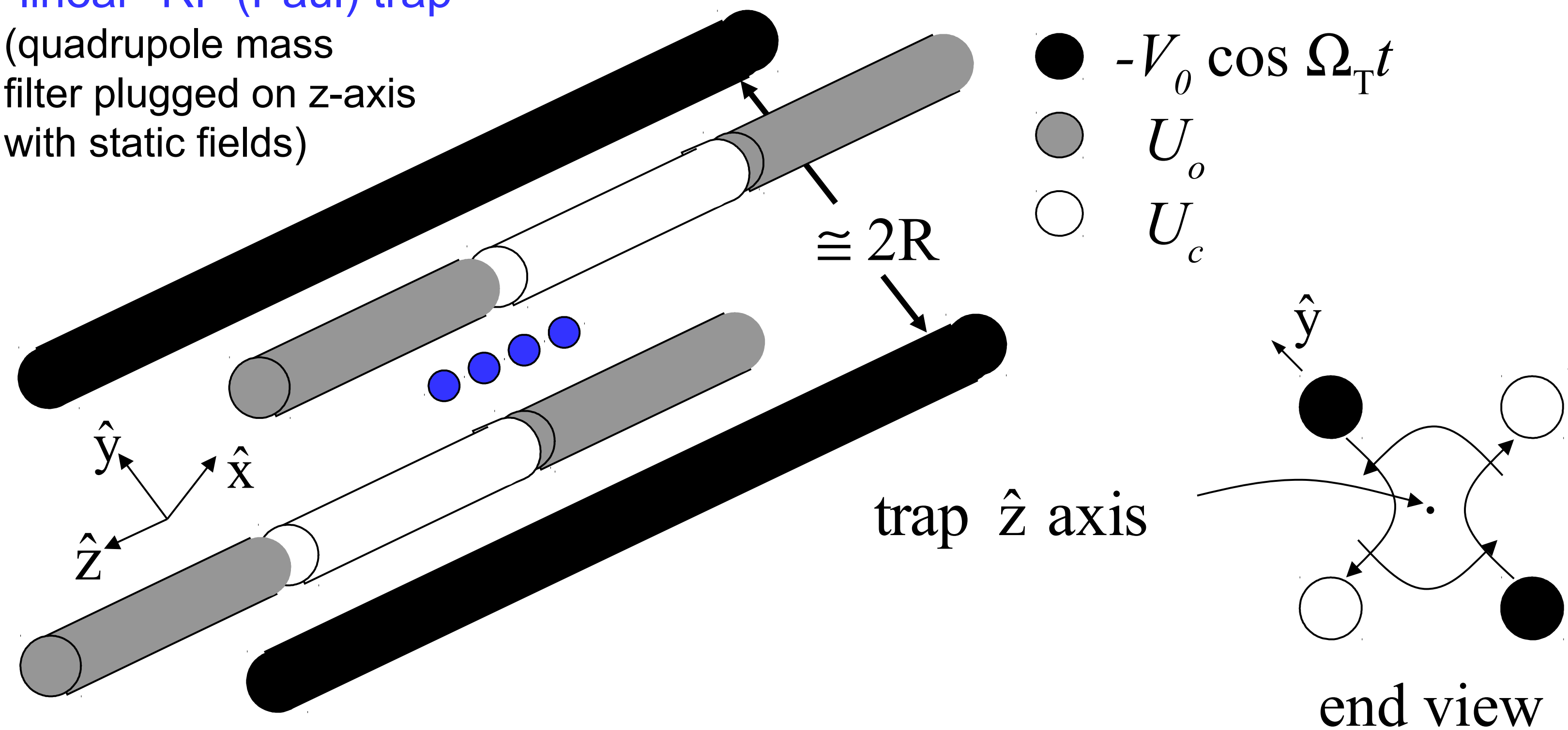
## Solution 2: RF-Paul trap:

$\Phi = [2z^2 - x^2 - y^2]\,(V_0\cos\Omega t + U_0)$ - establishes "pseudopotential"

Special case:

“linear” RF (Paul) trap

(quadrupole mass filter plugged on z-axis with static fields)




typically, RF frequency

near center of trap:

$$\Phi = \frac{(x^2 - y^2)}{2R^2} V_0 \cos \Omega_T t + \frac{(U_x x^2 + U_y y^2 + U_z z^2)}{2R^2},$$

Laplace:

$$\sum_{i=x,y,z} U_i = 0$$

$$U_x = \alpha_{xo} U_o + \alpha_{xc} U_c, \quad U_y = \alpha_{yo} U_o + \alpha_{yc} U_c, \quad U_z = \kappa (U_o - U_c)$$

(In practice, obtain α’s and κ numerically)

From previous page:

$$\Phi = \frac{(x^2 - y^2)}{2R^2} V_0 \cos \Omega_T t + \frac{(U_x x^2 + U_y y^2 + U_z z^2)}{2R^2}$$

Laplace:

$$\sum_{i=x,y,z} U_i = 0$$

Equations of motion (classical treatment adequate; quantum treatment: Wolfgang Schleich et al.)

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$$\vec{F} = m\vec{a} \Longrightarrow$$

$$\frac{d^2 x_i}{d\xi^2} + \left[a_i - 2q_i \cos 2\xi\right] x_i = 0 \qquad (i \in \{x, y, z\}) \quad (2)$$

Mathieu equation

$$a_i \equiv 4qU_i / m\Omega_T^2 R^2, \quad \xi \equiv \Omega_T t/2$$

$$q_x = -q_y \equiv -2qV_0 / m\Omega_T^2 R^2, \quad q_z = 0$$

z-motion, $q_z = 0$ (static harmonic well)

$$\omega_z = (qU_z / mR^2)^{1/2} = \sqrt{a_z}\,\Omega_T / 2$$

x,y motion, Mathieu equation:

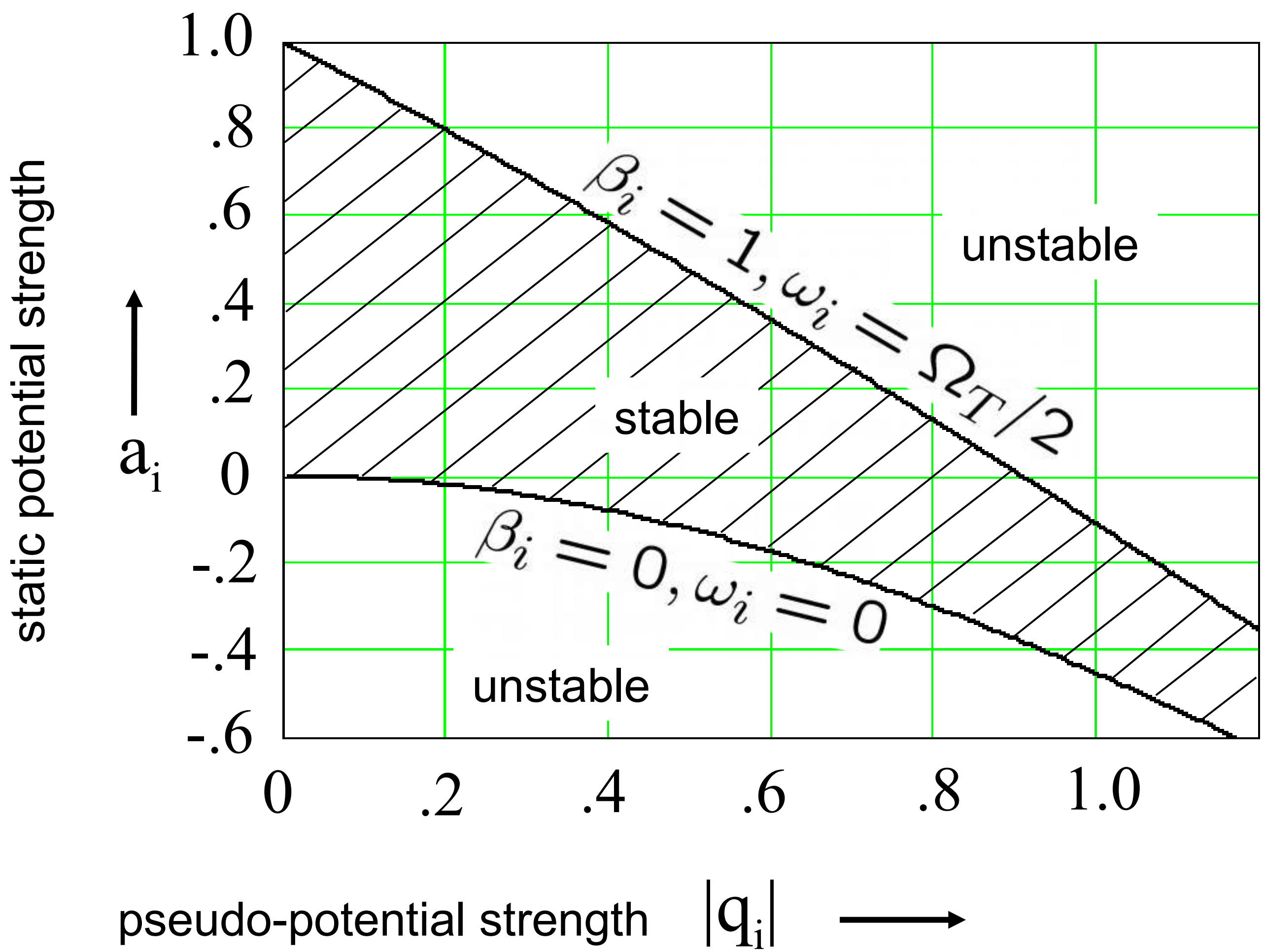
$$x_i(\xi) = Ae^{i\beta_i \xi} \sum_{n=-\infty}^{\infty} C_{2n} e^{i2n\xi} + Be^{-i\beta_i \xi} \sum_{n=-\infty}^{\infty} C_{2n} e^{-i2n\xi} \qquad (i \in \{x, y\})$$

plug into (2), find (recursion relation for) $C_{2n}$

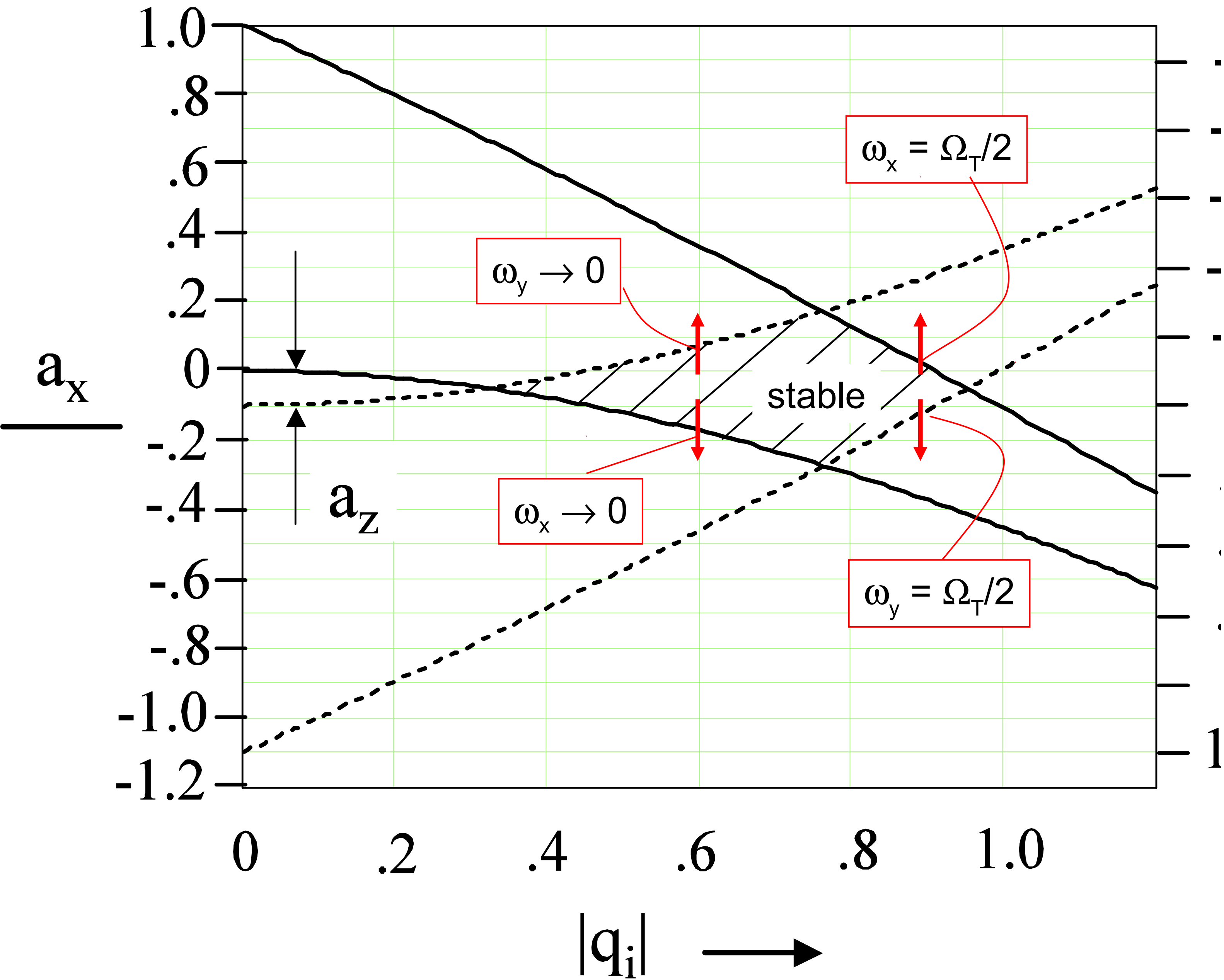
Solution in i[th] direction ($i \in \{x,y\}$):

$$a_i \equiv 4qU_i/m\Omega_T^2 R^2$$
$$q_x = -q_y \equiv -2qV_0/m\Omega_T^2 R^2$$

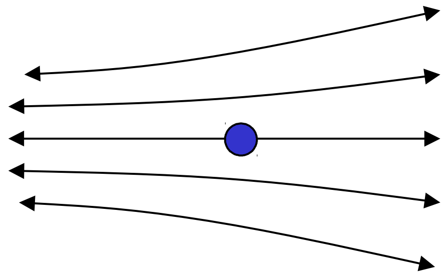
$\omega_i$ =
ponderomotive potential
binding frequency



Simultaneous solution for x, y, z: $a_i \propto U_i$, $a_x + a_y + a_z = 0$



# Heuristic approach: "pseudo-potential" approximation



$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})\cos\Omega_T t$$

- <u>assume</u> mean ion position changes negligibly in duration $2\pi/\Omega_T$

$$m\frac{\partial^2 \vec{x}_\mu}{\partial t^2} = q\vec{E}(\vec{x})\cos\Omega_T t \quad \Rightarrow \quad \vec{x}_\mu = -\frac{q\vec{E}(\vec{x})}{m\Omega_T^2}\cos\Omega_T t \quad \text{"micromotion"}$$

- pseudo-potential from micromotion kinetic energy:

$$U(\vec{x})_{pseudo} = \langle KE(\vec{x})\rangle = \frac{1}{2} m\langle v(x)_\mu^{\,2}\rangle = \frac{q^2 E^2(\vec{x})}{4m\Omega_T^2}$$

Or, calculate avg. force $\langle\vec{F}\rangle$ over one RF cycle. Then $U_{pseudo} = -\int\langle\vec{F}\rangle\cdot d\vec{x}$

- for linear RF trap,

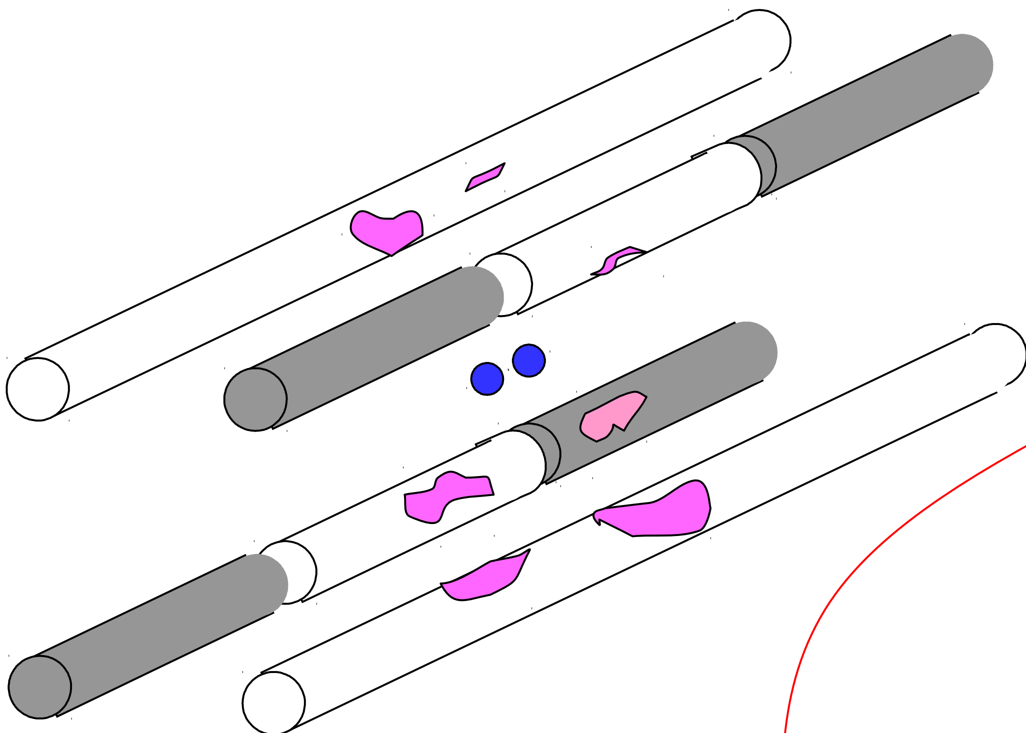
$$U_{pseudo} = \frac{q^2 V_0^2}{2m^2\Omega_T^2 R^4}(x^2+y^2) = \frac{1}{2}m\omega_{x,y}^2(x^2+y^2) \qquad \omega_{x,y} = \frac{qV_0}{\sqrt{2}m\Omega_T R^2}$$

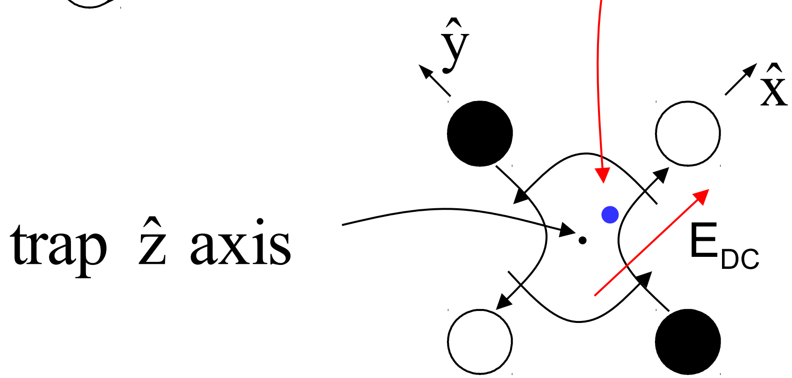
including perturbation from micromotion:

$$x_i(t) \simeq X_{i0}\cos\omega_i t + \left[X_{i0}\cos\omega_i t\right]\frac{q_i}{2}\sin\Omega_T t$$

agrees with Mathieu equation in limit $|a_i|$, $q_i^2 << 1$

# (some) ion-trap realities






electrode surface
patch potentials:

Static potential (e.g. static patch field): pushes ions away from trap axis
⇒ micromotion $x_\mu \sin \Omega_T t$

Fluctuating patch fields: causes heating of ion motion (at $\omega_i$)

Source: unknown!
(mobile electrons or adsorbates on oxide layers,….. ??)

$$x_\mu = X_{displacement} \frac{q_i}{2} \sin \Omega_T t$$

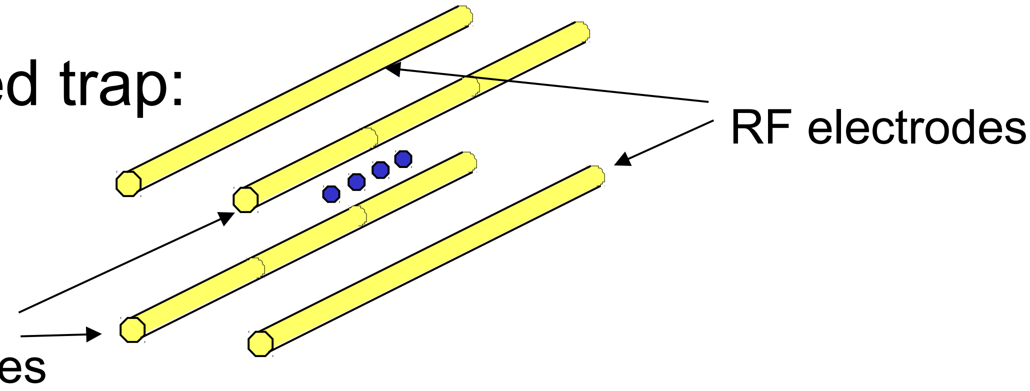
recall: $x_i(t) \simeq X_{i0} \cos \omega_i t + \left[X_{i0} \cos \omega_i t\right] \frac{q_i}{2} \sin \Omega_T t$

# Trap fabrication

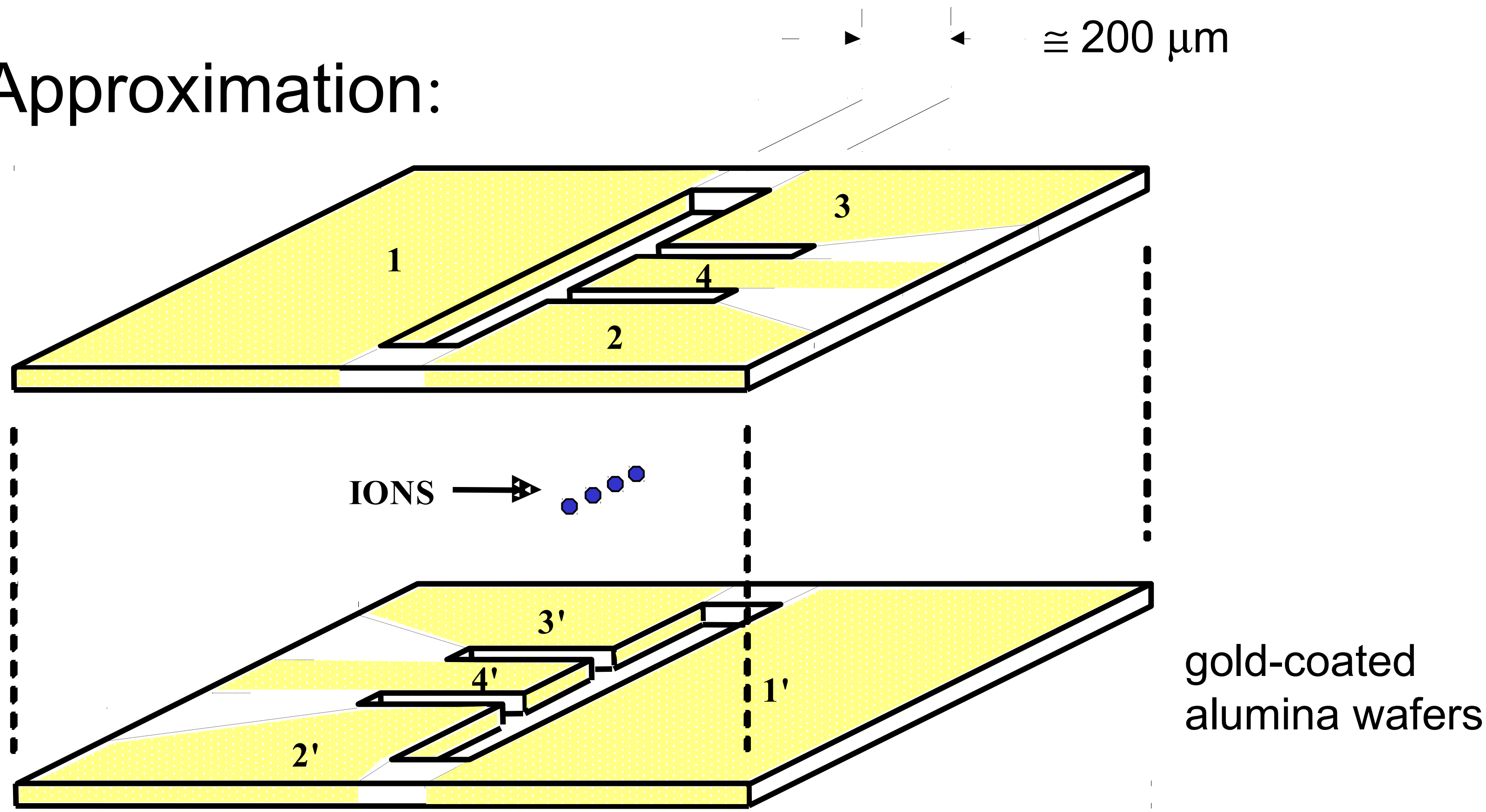
(Gate speed $\propto \omega_{motion}$ $\propto$ (dimensions)$^{-2}$),

$\Rightarrow$ want trap small

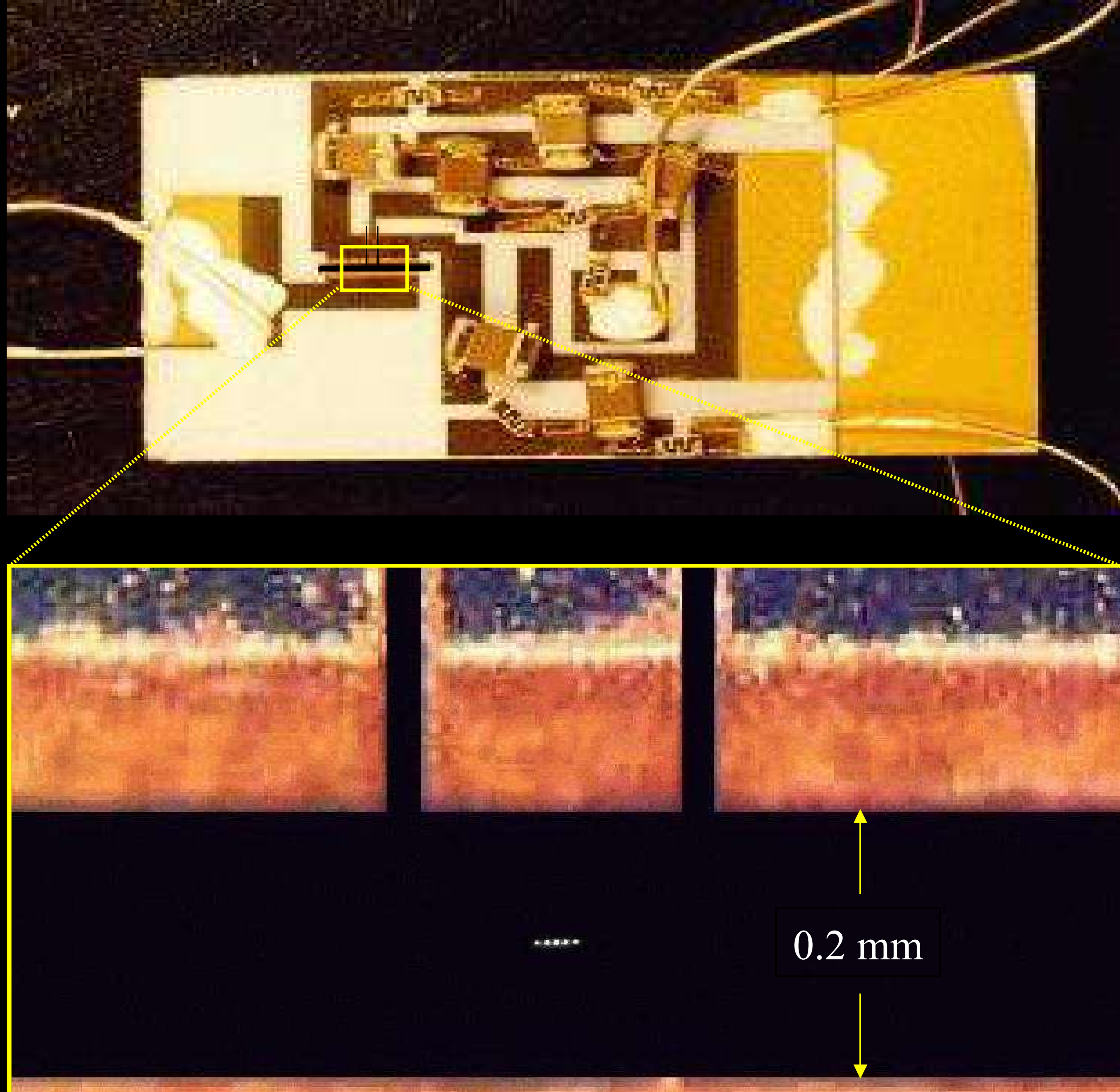
## idealized trap:




## Approximation:

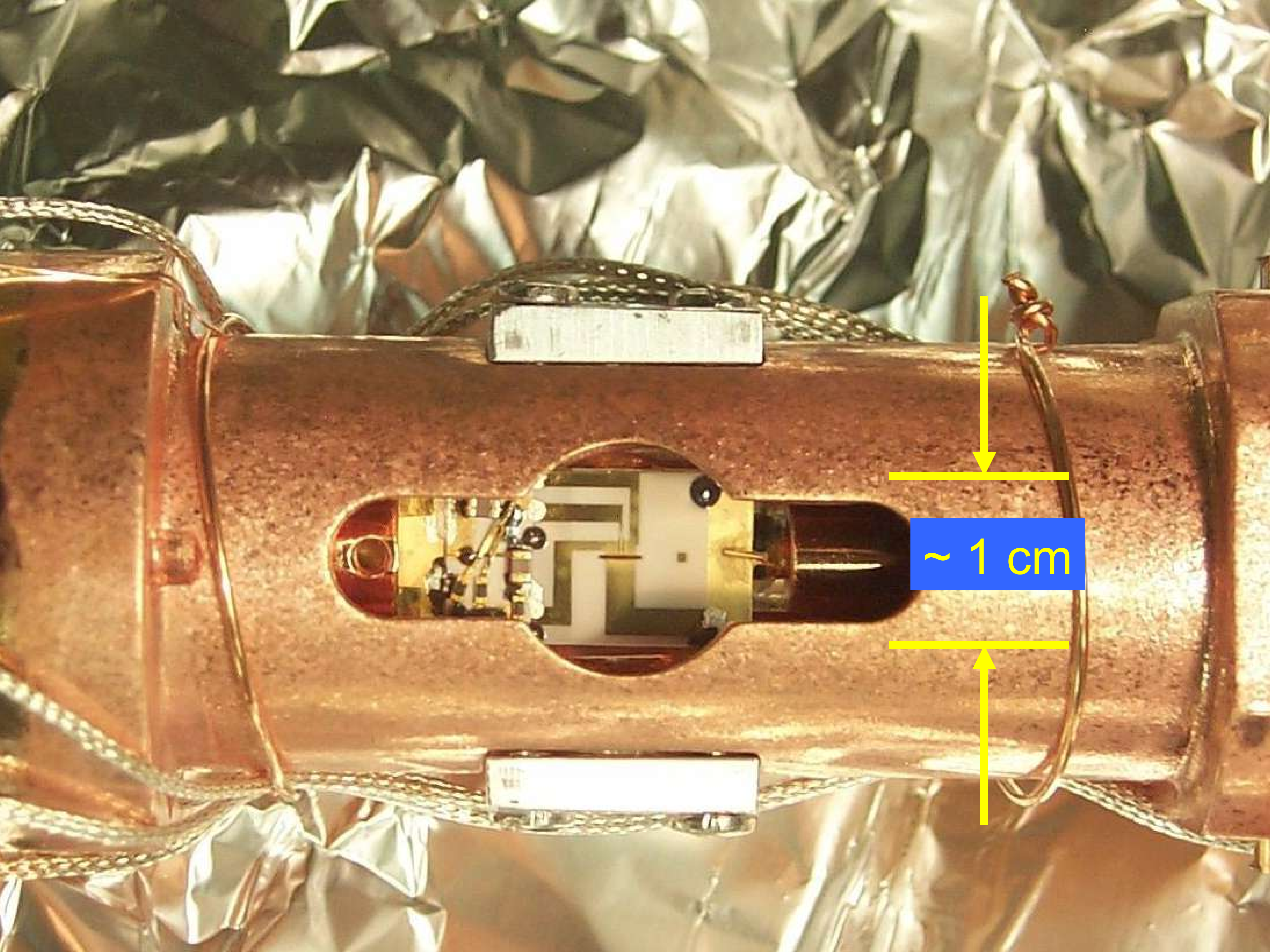


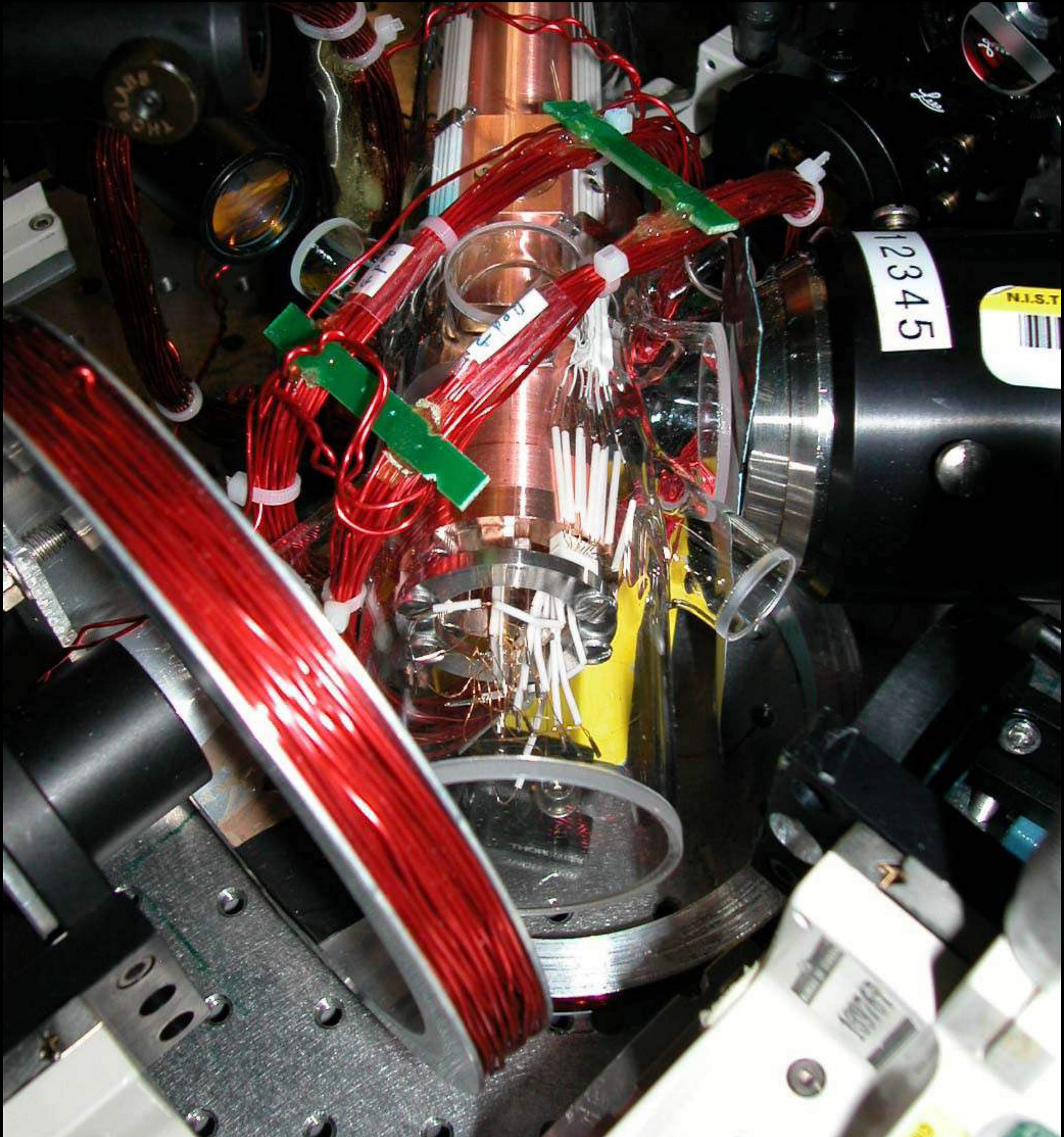

M. Rowe *et al.*, Quant. Inform. Comp. **2**, 257 (2002).




For $^9Be^+$, $V_0$ = 500 V, $\Omega_T/2\pi$ = 200 MHz, R ≈ 200 μm

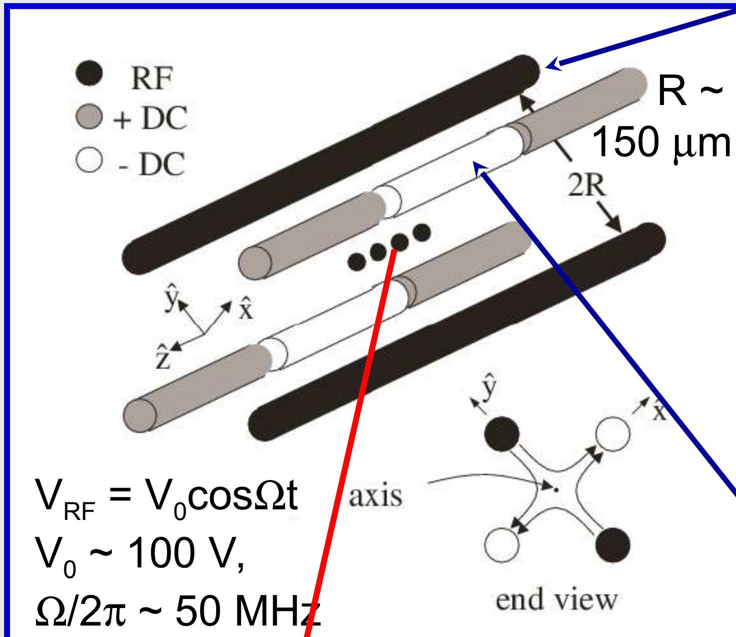
$\omega_{x,y}/2\pi$ ~ 6 MHz, $q_{x,y}$ ~ 0.085

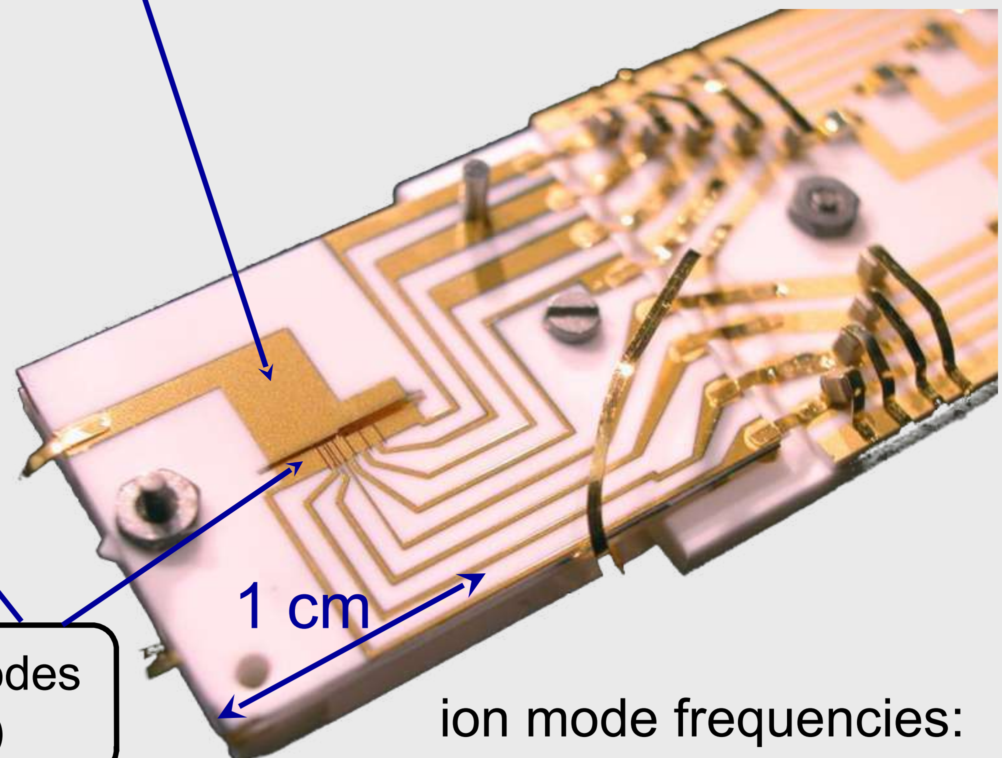

~ 1 cm


12345
N.I.S.T

- Multizone linear trap
  - Gold on alumina
  - Two wafers

rf electrodes

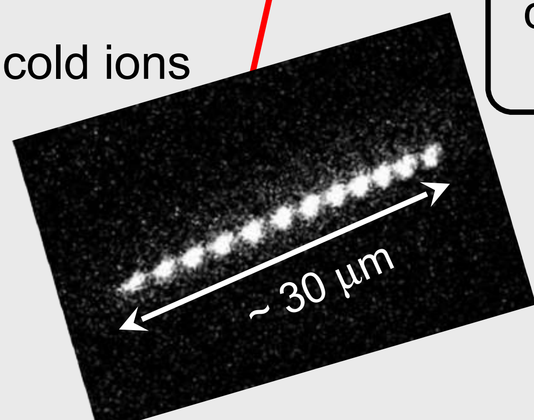



control electrodes (rf ground)

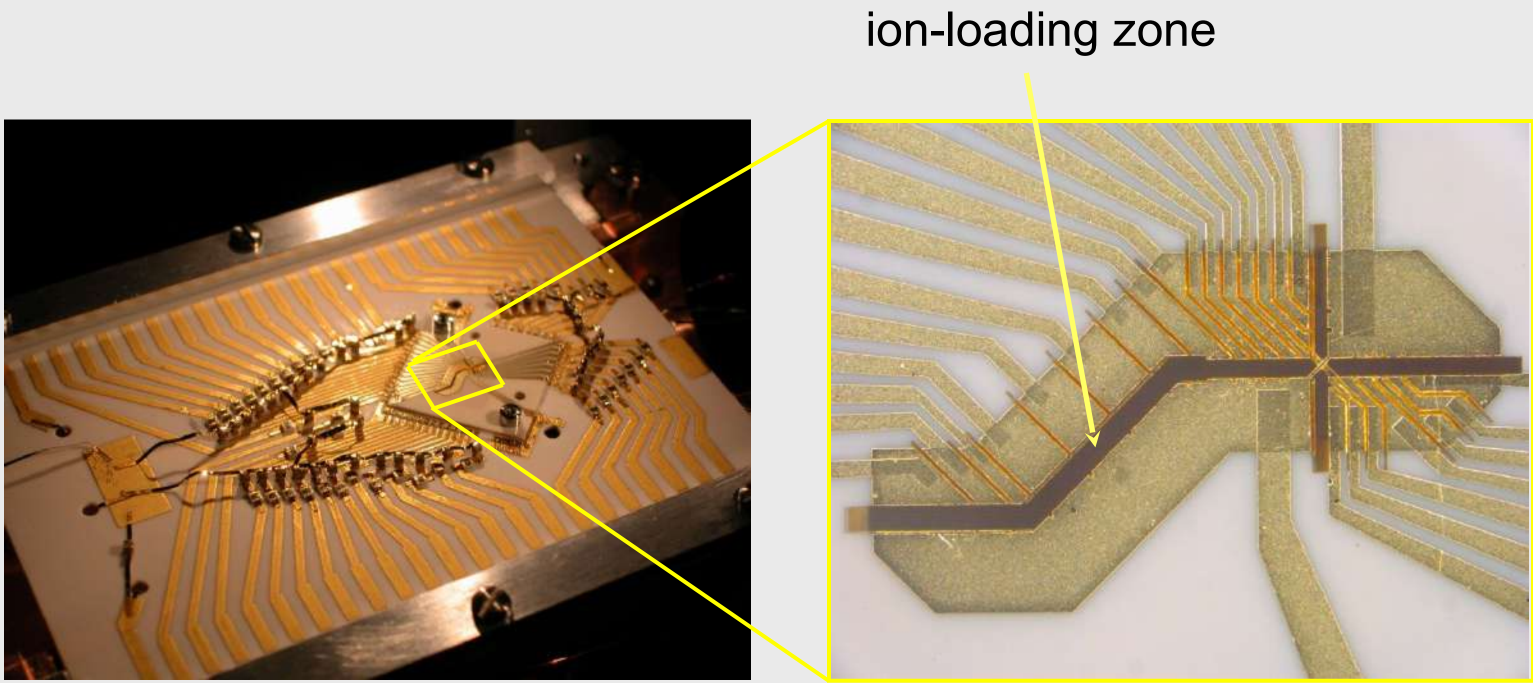
ion mode frequencies:
$\omega/2\pi \cong$ 1-10 MHz

cold ions




- Address individual ions with focused laser beams

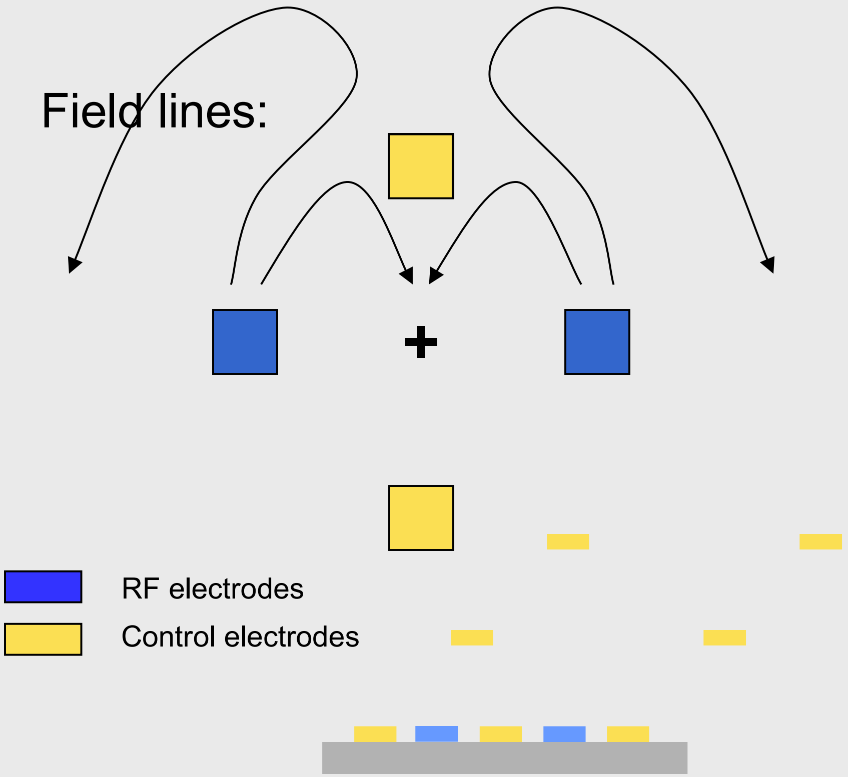
# Further scaling: 2-D traps

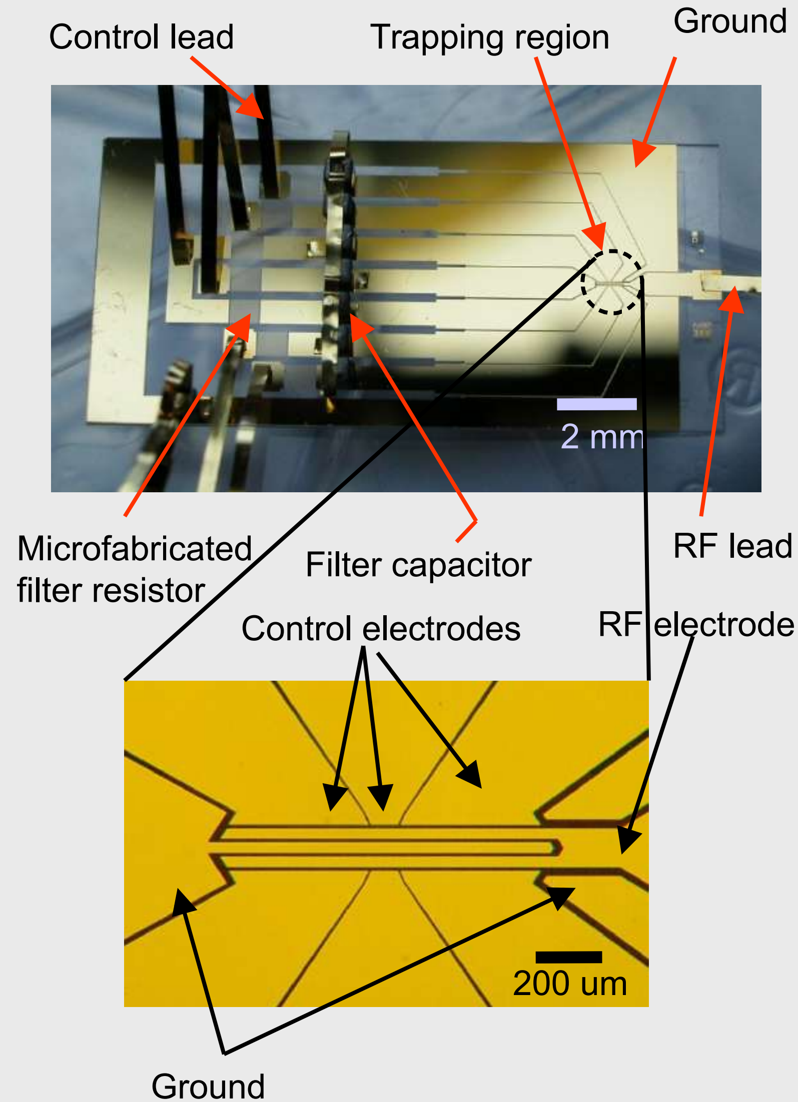



NIST, Au on $Al_2O_3$ substrate
2-layer, 18 zones
R. B. Blakestad et al., Phys. Rev. A**84**, 032314 (2011)

# Surface-electrode trap

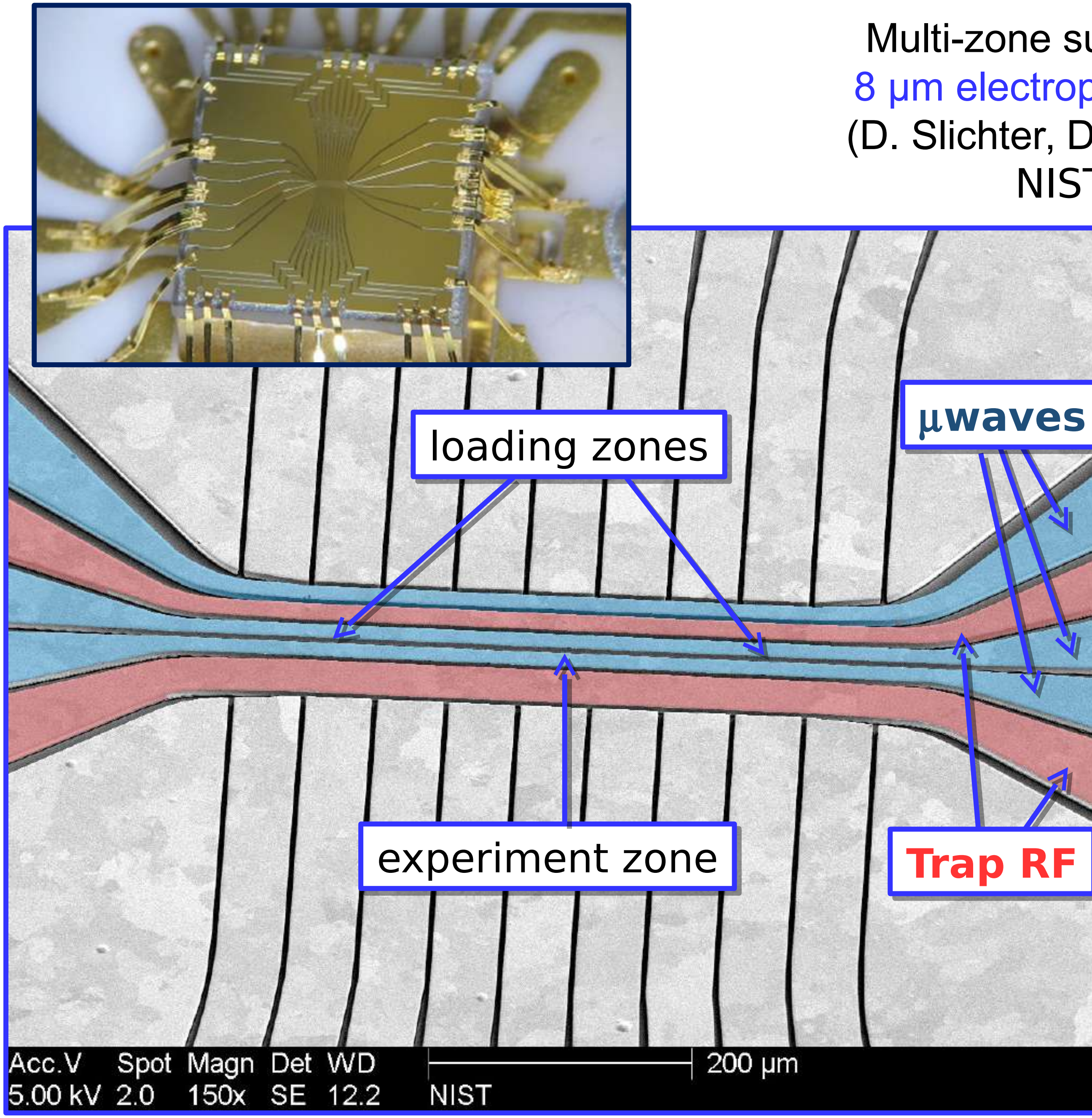
(easier to make small trap electrodes)




S. Seidelin et al. Phys. Rev. Lett. **96**, 253003 (2006)

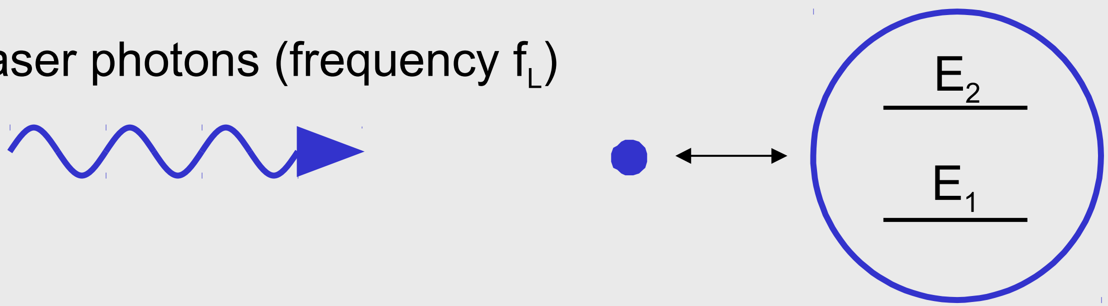
Multi-zone surface-electrode trap
8 μm electroplated Au on sapphire
(D. Slichter, D. Allcock, R. Srinivas
NIST, Boulder)

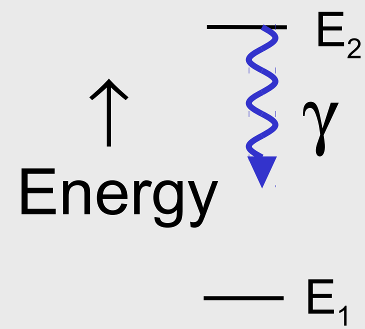



- ion height above surface = 30 μm
- 3 microwave lines ⇒ large ∇B, B ≅ 0 at ion position (for magnetic-field-induced multiqubit gates)
- loading zones for reduced stray fields
- can eliminate need for high-power lasers for logic gates (used in quantum-logic detection)

# Doppler laser cooling:

Laser photons (frequency $f_L$)



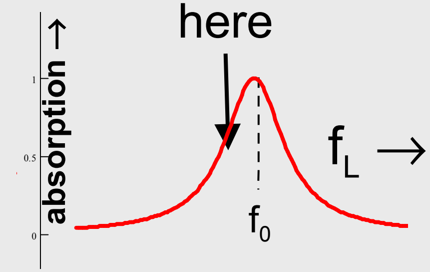

1. quantum mechanics ⇒ atoms exist in discrete energy states

2. atoms at rest absorb and re-emit photons maximally when laser frequency $f_L = f_0 = (E_2 - E_1)/h$, (h = Planck's constant)

3. Doppler shift: moving atoms absorb radiation maximally when laser frequency $f_L = f_0(1 - v/c)$ (c = speed of light)


4. momentum of absorbed photons reduces atom's momentum (and velocity) ⇒ cooling!

for best cooling, tune laser here




$\langle E_{kinetic} \rangle_{min} \cong \hbar\gamma/4$

$(T \cong 0.001\ \mathrm{K})$

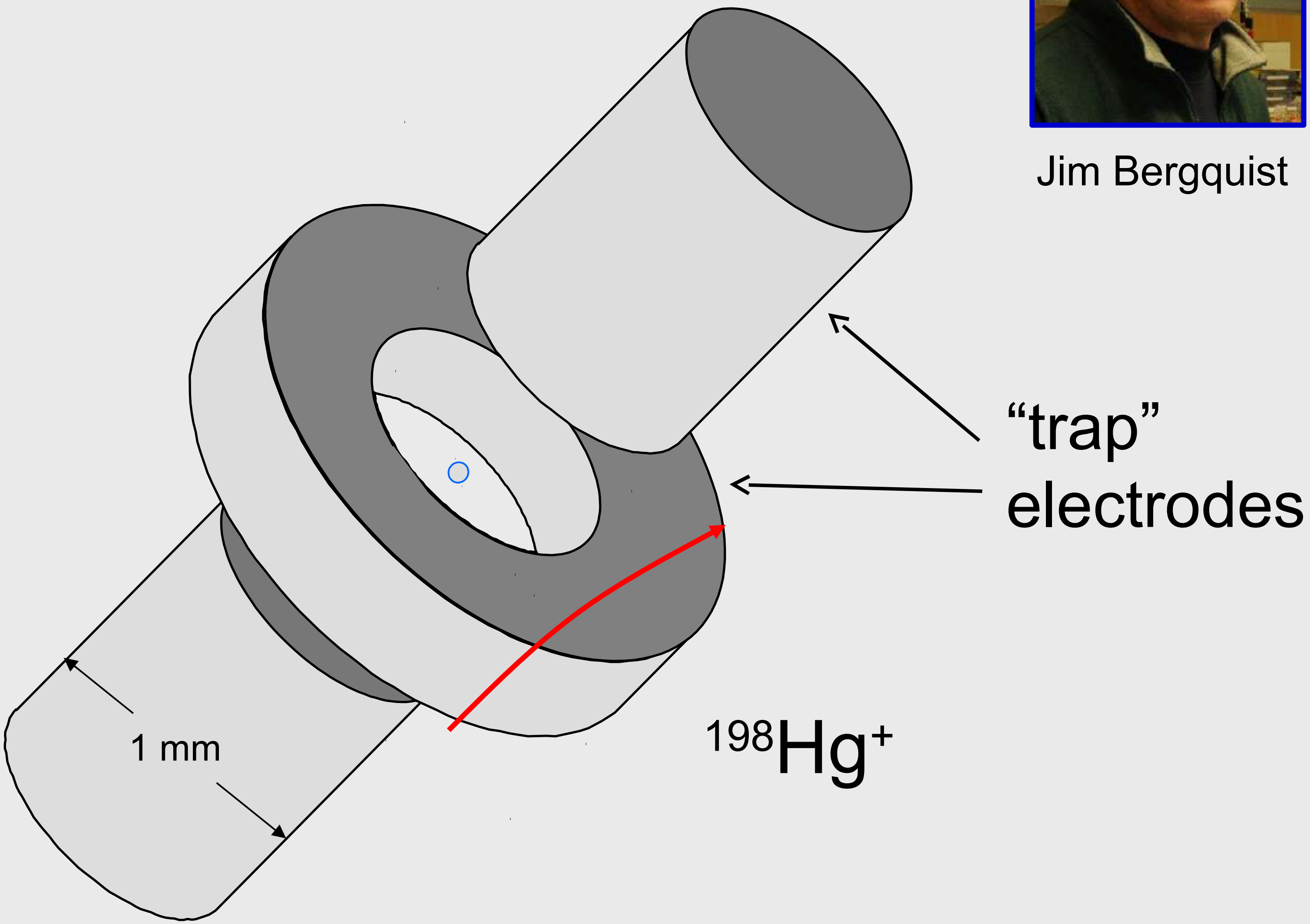

D. J. Wineland and H. Dehmelt, Bulletin, Am. Phys. Soc. **20**, 637 (1975)

T. W. Hänsch and A. L. Schawlow, Opt. Comm. **13**, 68 (1975)

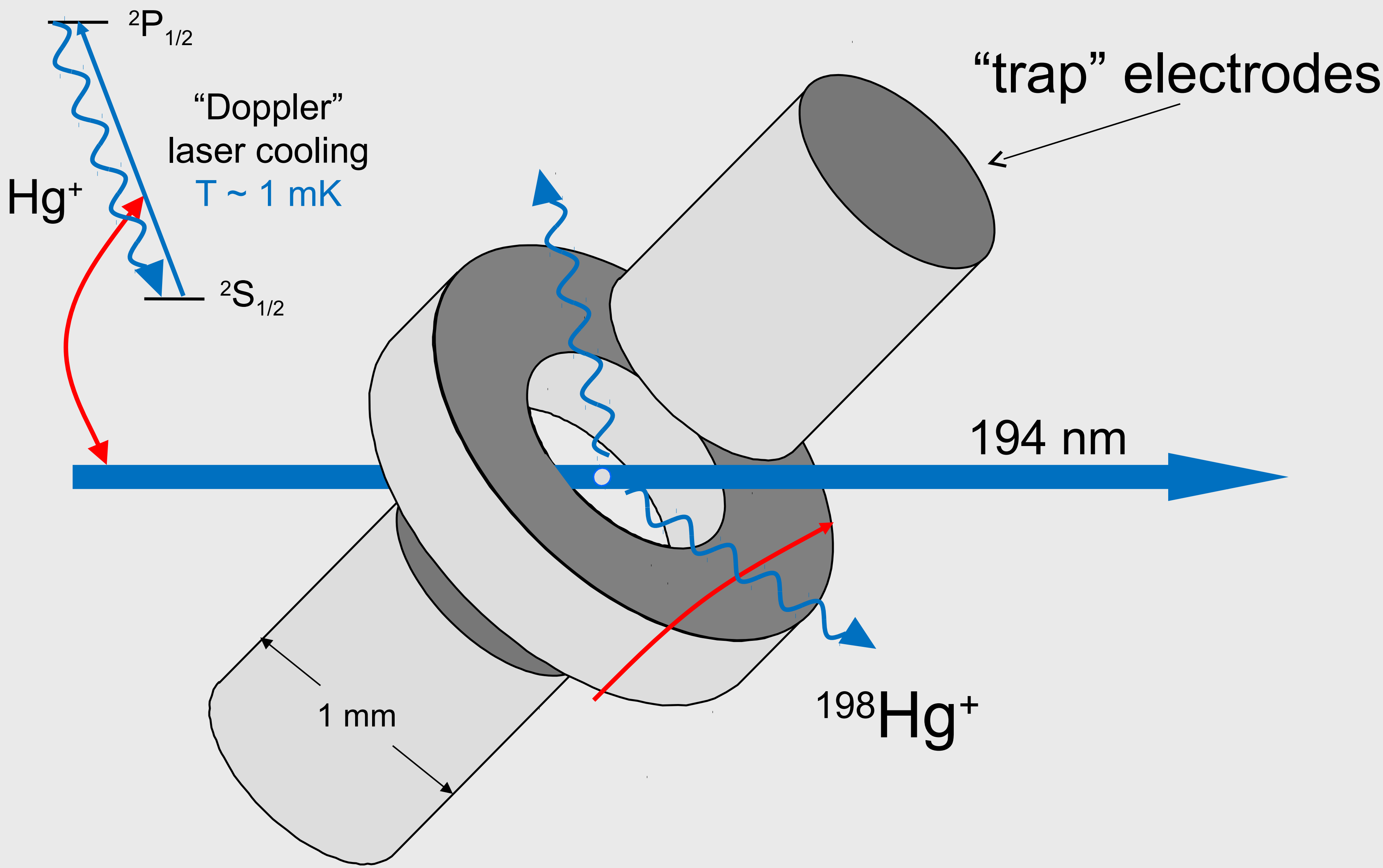
Example: mercury ion ($Hg^+$) experiments (NIST, Boulder)

Jim Bergquist

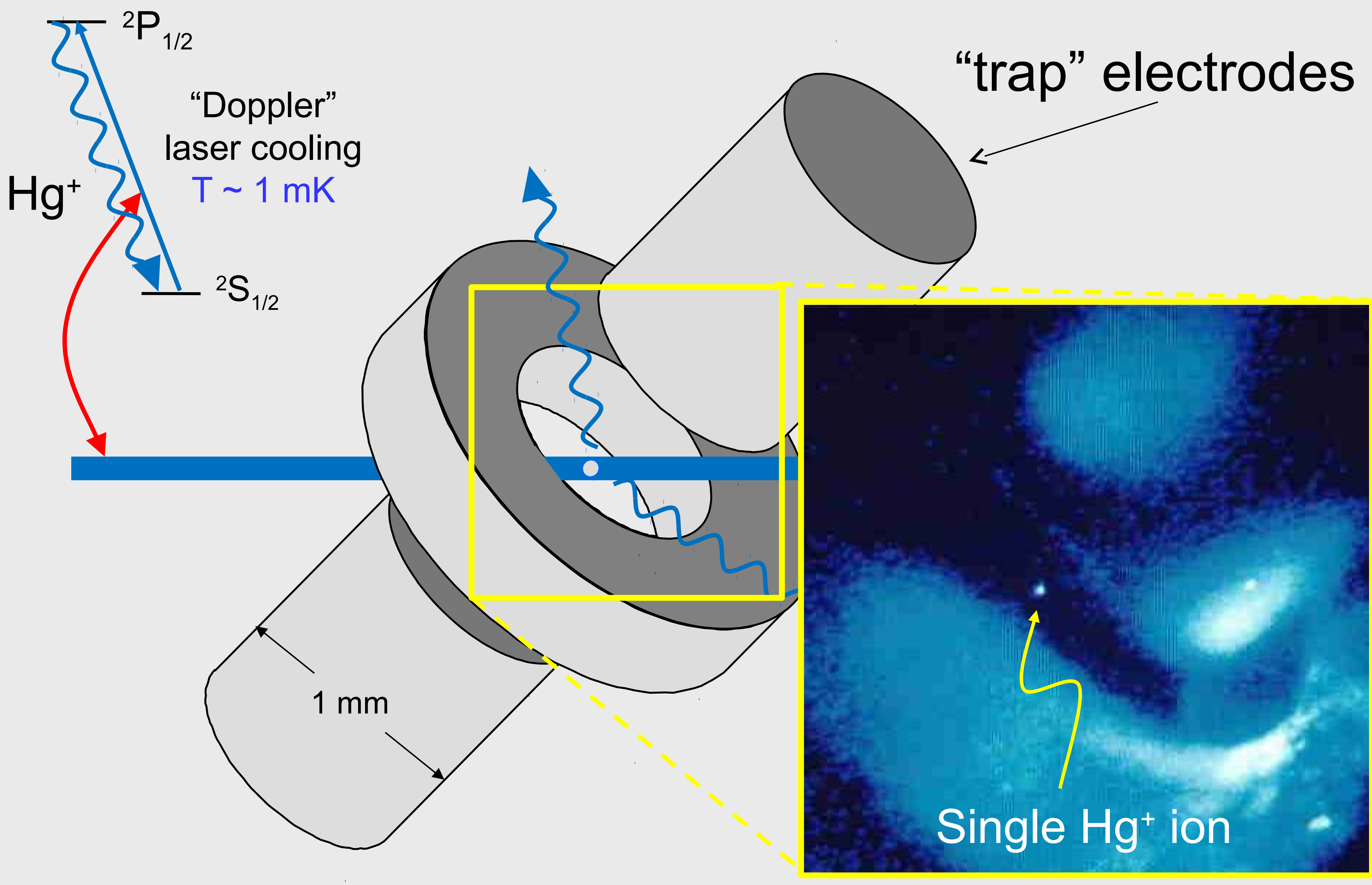
(simple) RF Paul trap:



# Mercury ion ($Hg^+$) experiments at NIST



# Mercury ion ($Hg^+$) experiments at NIST



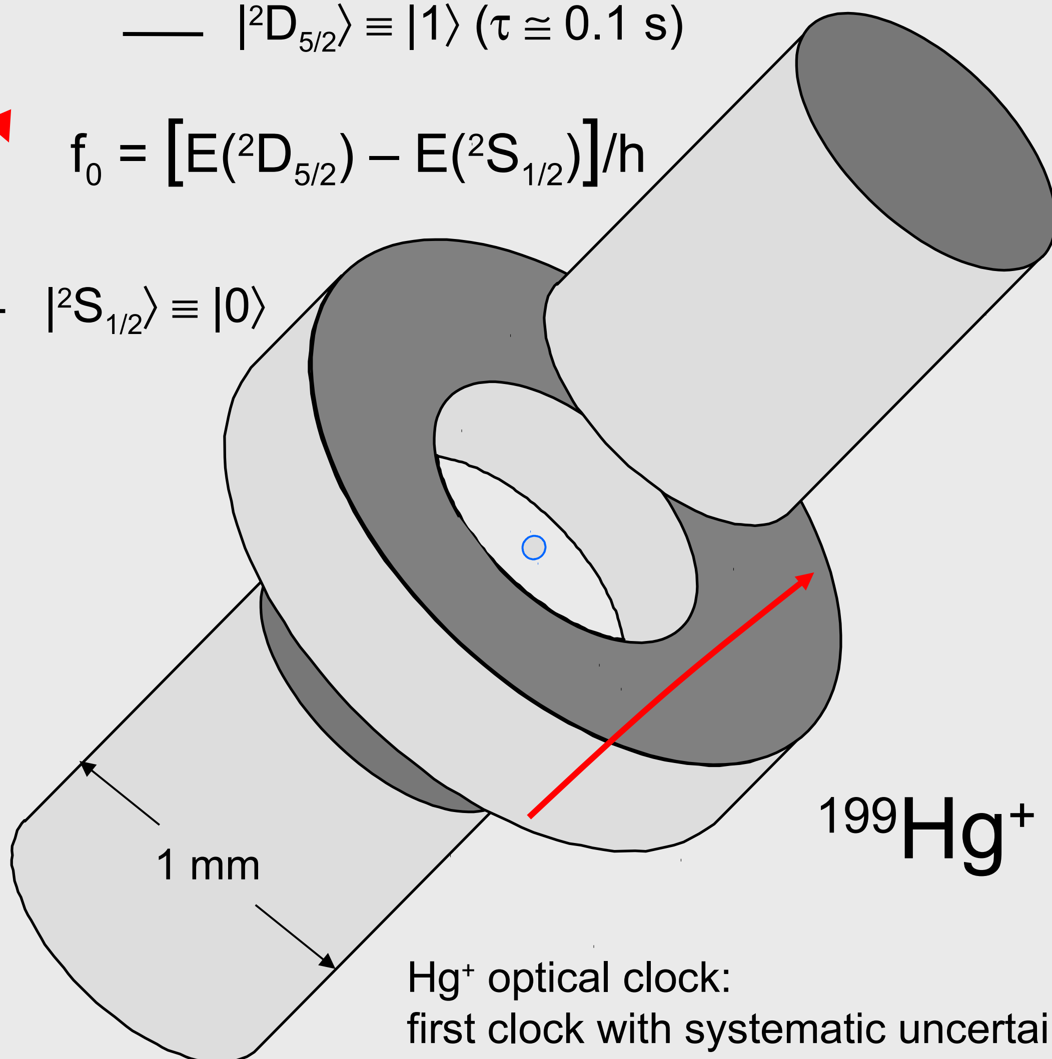
# Mercury ion clock experiments: J. C. Bergquist et al.

$f_0$ = 1 064 721 609 899 144.94 (97) Hz (2006)

$|^2D_{5/2}\rangle \equiv |1\rangle$ ($\tau \cong 0.1$ s)

$$f_0 = \left[E(^2D_{5/2}) - E(^2S_{1/2})\right]/h$$

$|^2S_{1/2}\rangle \equiv |0\rangle$

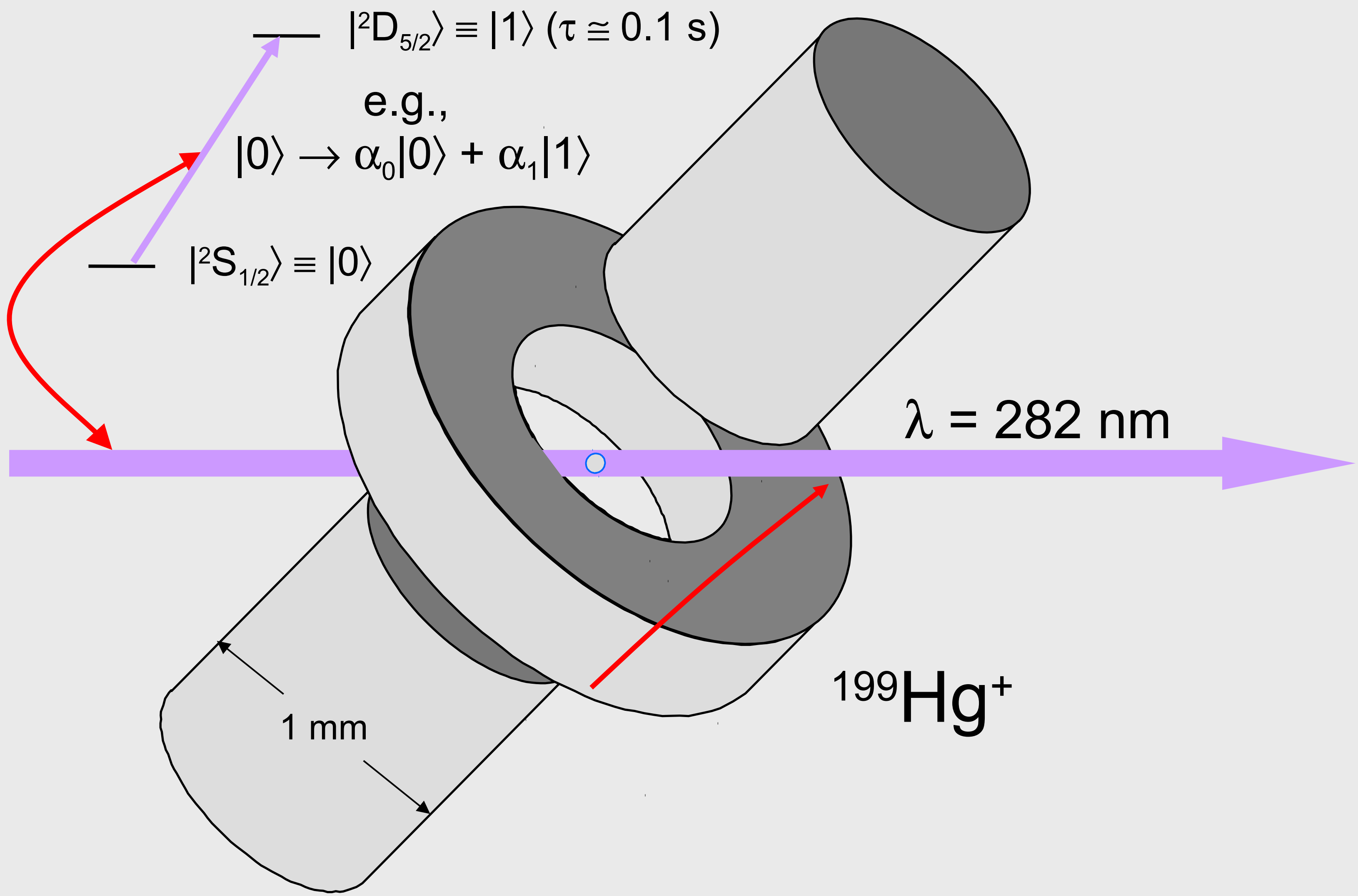



$^{199}Hg^+$

$Hg^+$ optical clock:
first clock with systematic uncertainly ($7 \times 10^{-17}$) below Cesium
W. H. Oskay et al., Phys. Rev. Lett. **97**, 020801 (2006)

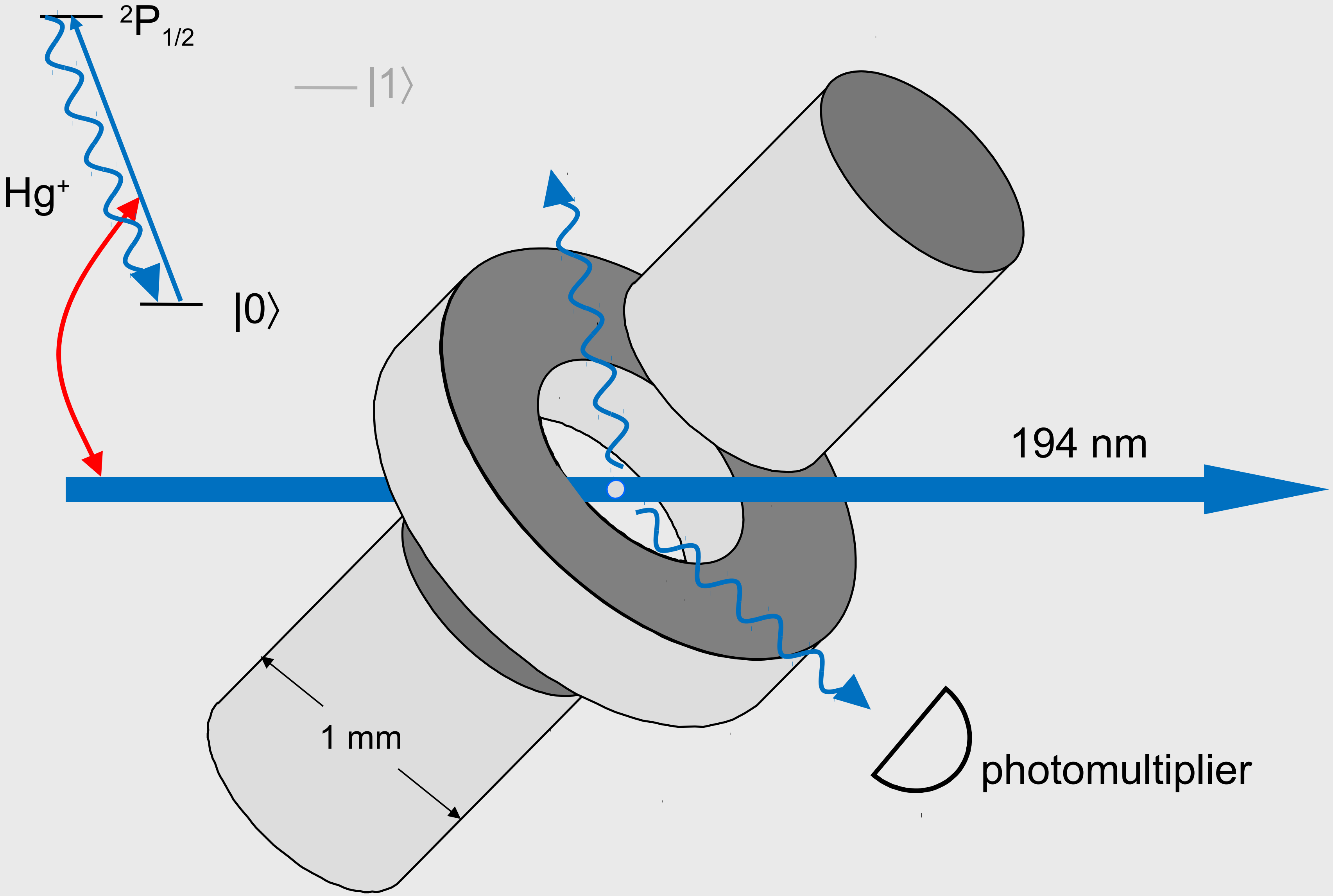
# Mercury ion quantum bit (“qubit”) experiments at NIST

superposition of “internal” energy states of ion



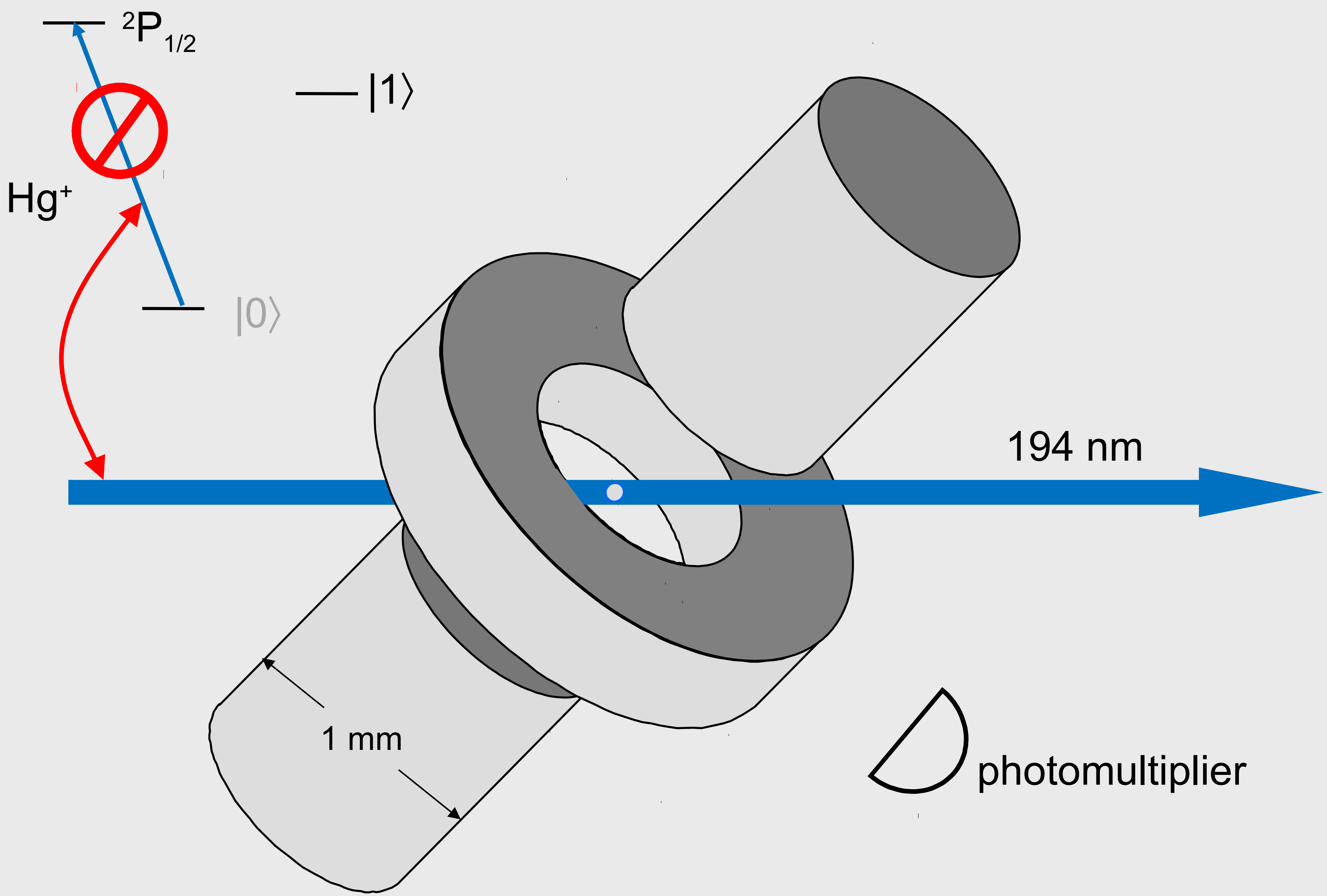
measurement of mercury ion qubit superposition

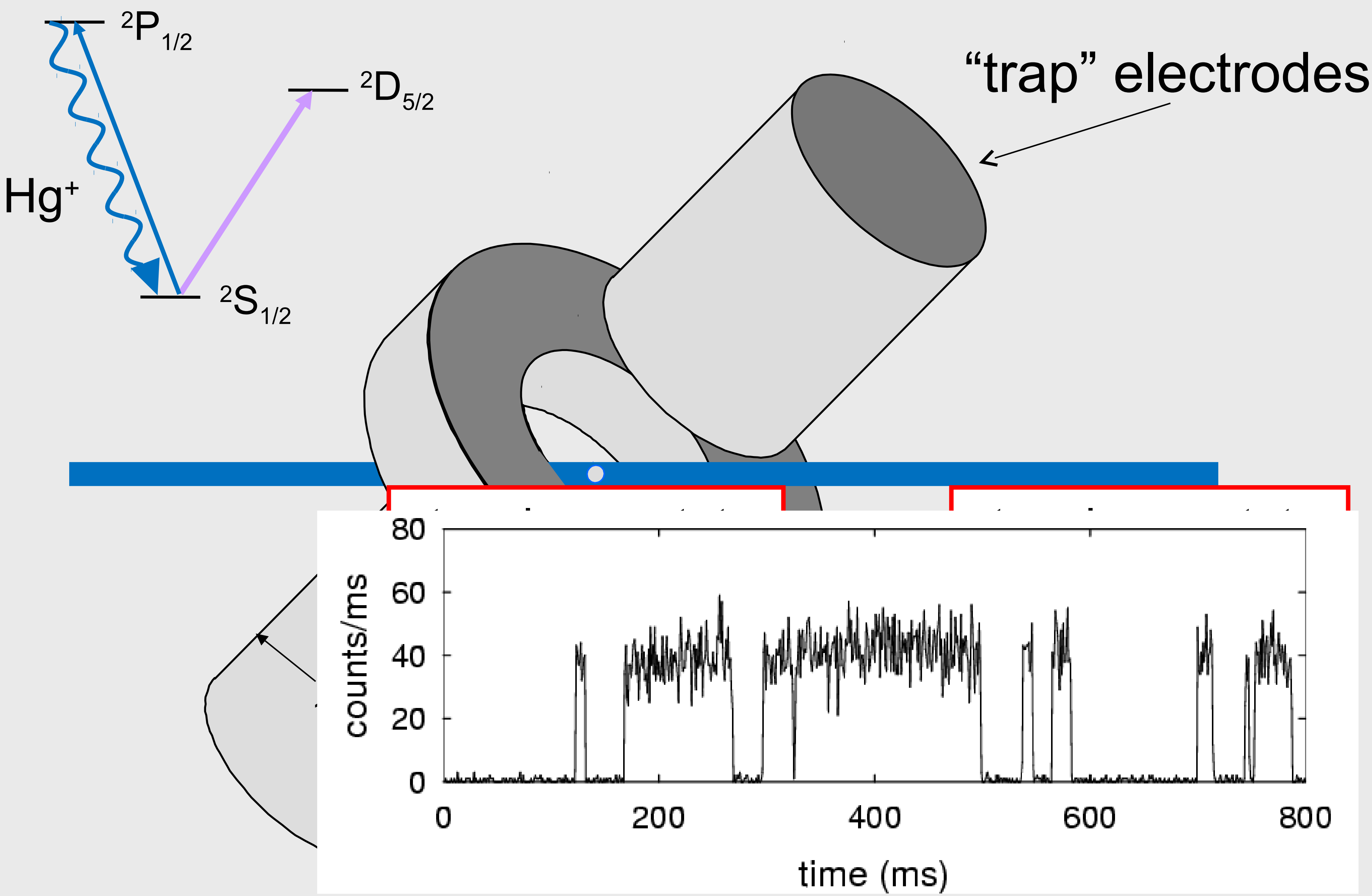
$\alpha_0|0\rangle + \alpha_1|1\rangle \to |0\rangle$



# measurement of mercury ion qubit superposition

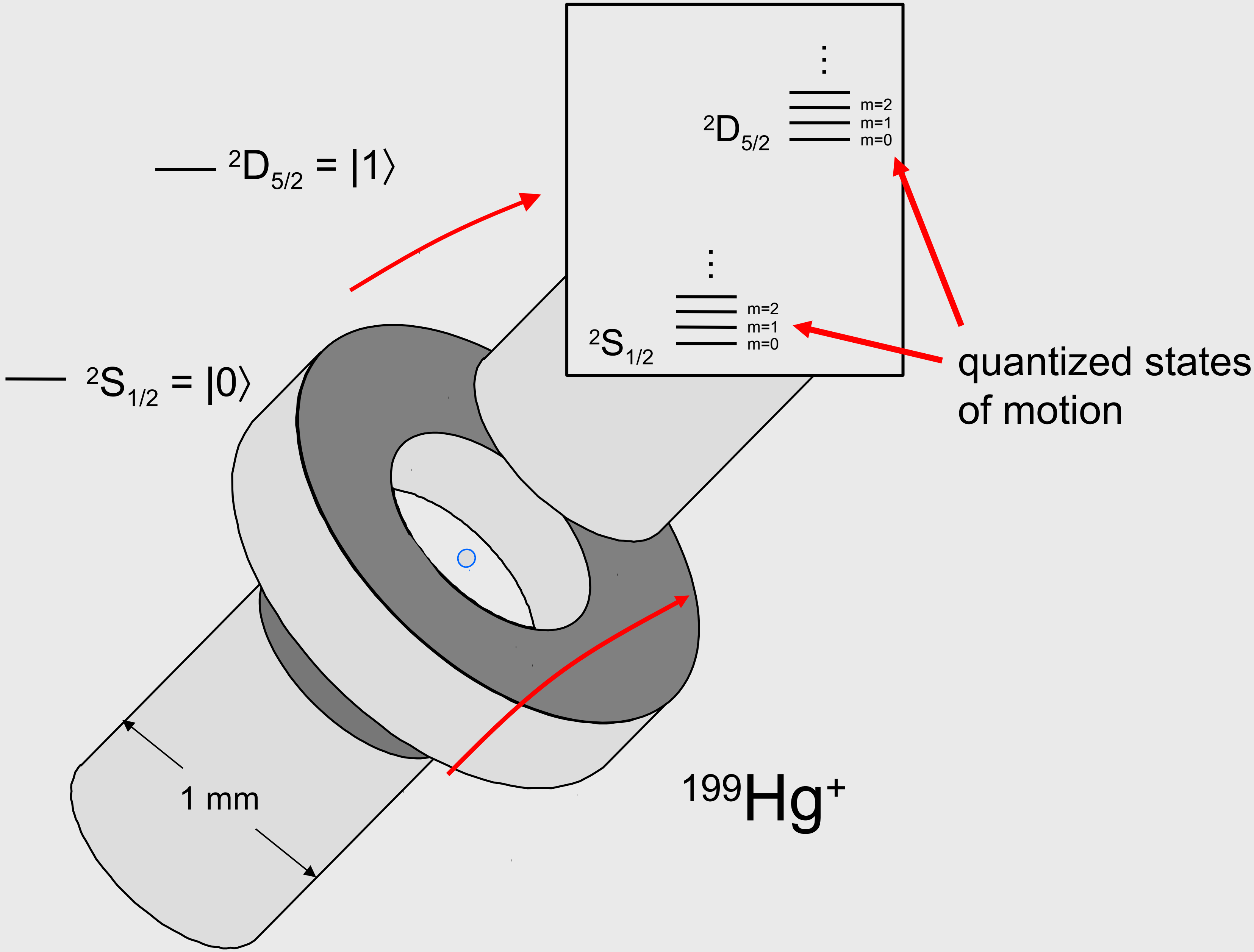
or: $\alpha_0|0\rangle + \alpha_1|1\rangle \rightarrow |1\rangle$



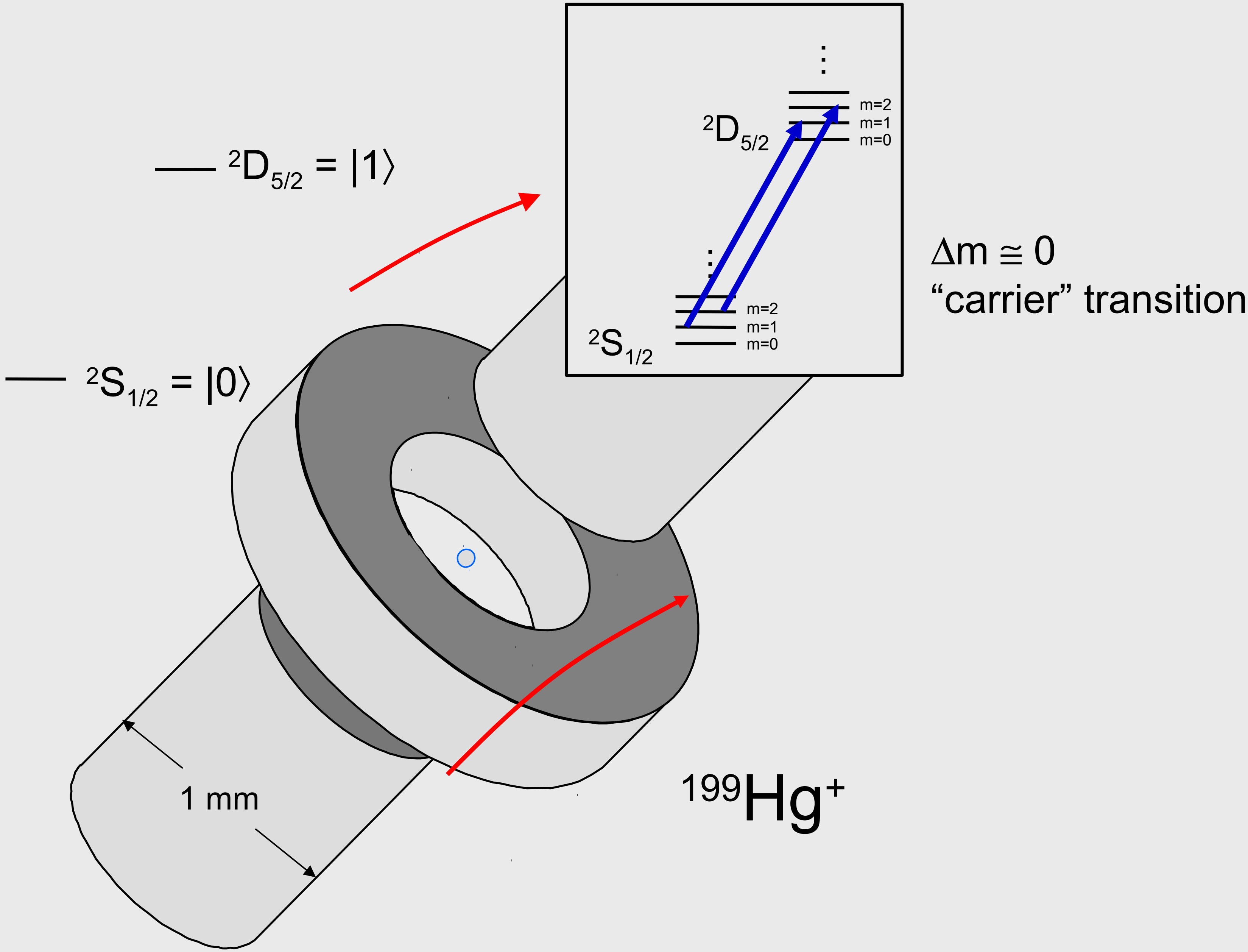


$^2P_{1/2}$
$^2D_{5/2}$
$Hg^+$
$^2S_{1/2}$
“trap” electrodes
counts/ms
80
60
40
20
0
0
200
400
600
800
time (ms)

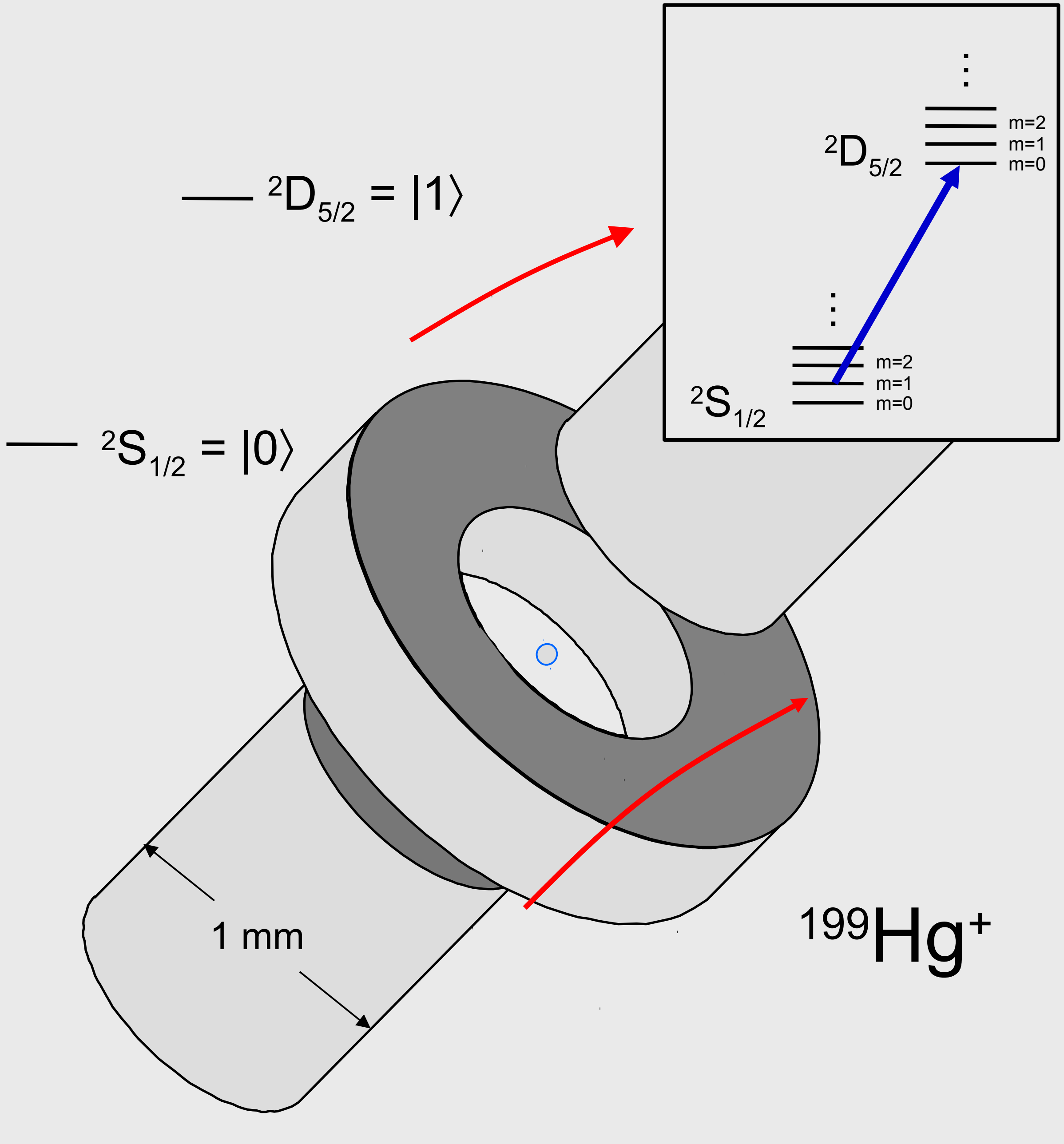
# fine-scale energy structure:



# fine-scale energy structure:

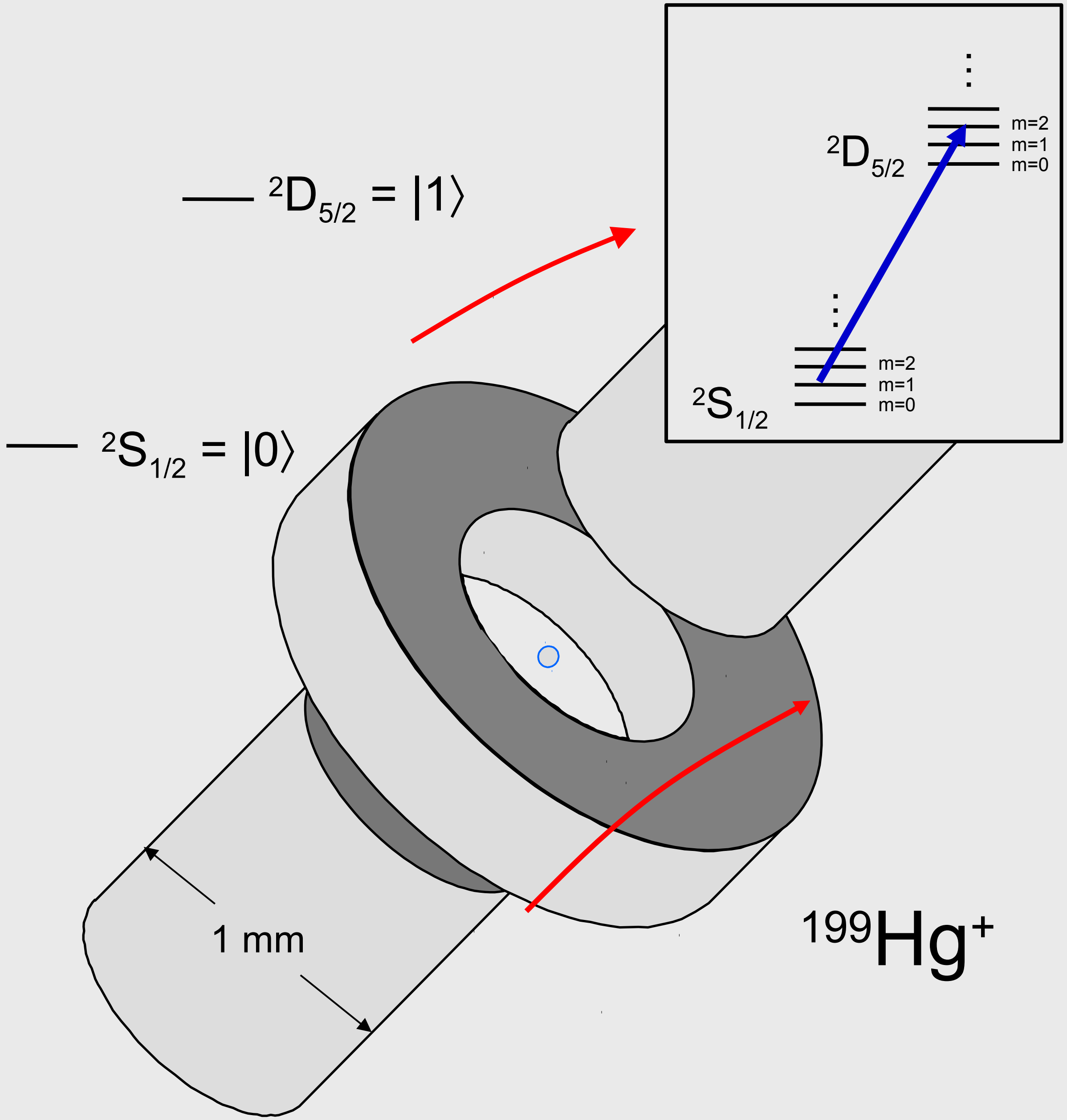


# fine-scale energy structure:




reduce
motional state
by one quantum
$\Delta m = -1$
“red sideband”
transition

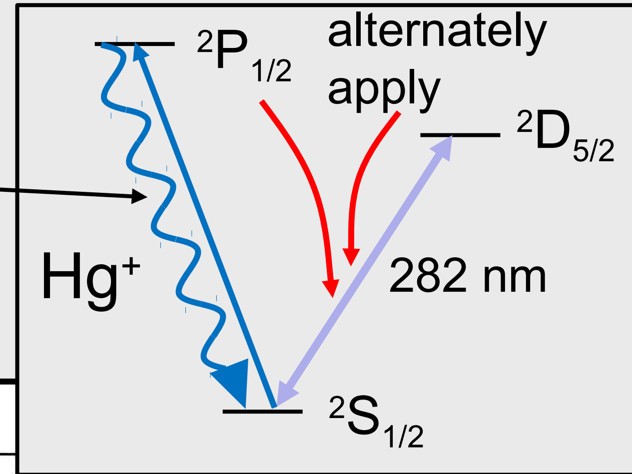
# fine-scale energy structure:




increase motional state by one quantum
$\Delta m = +1$
"blue sideband" transition

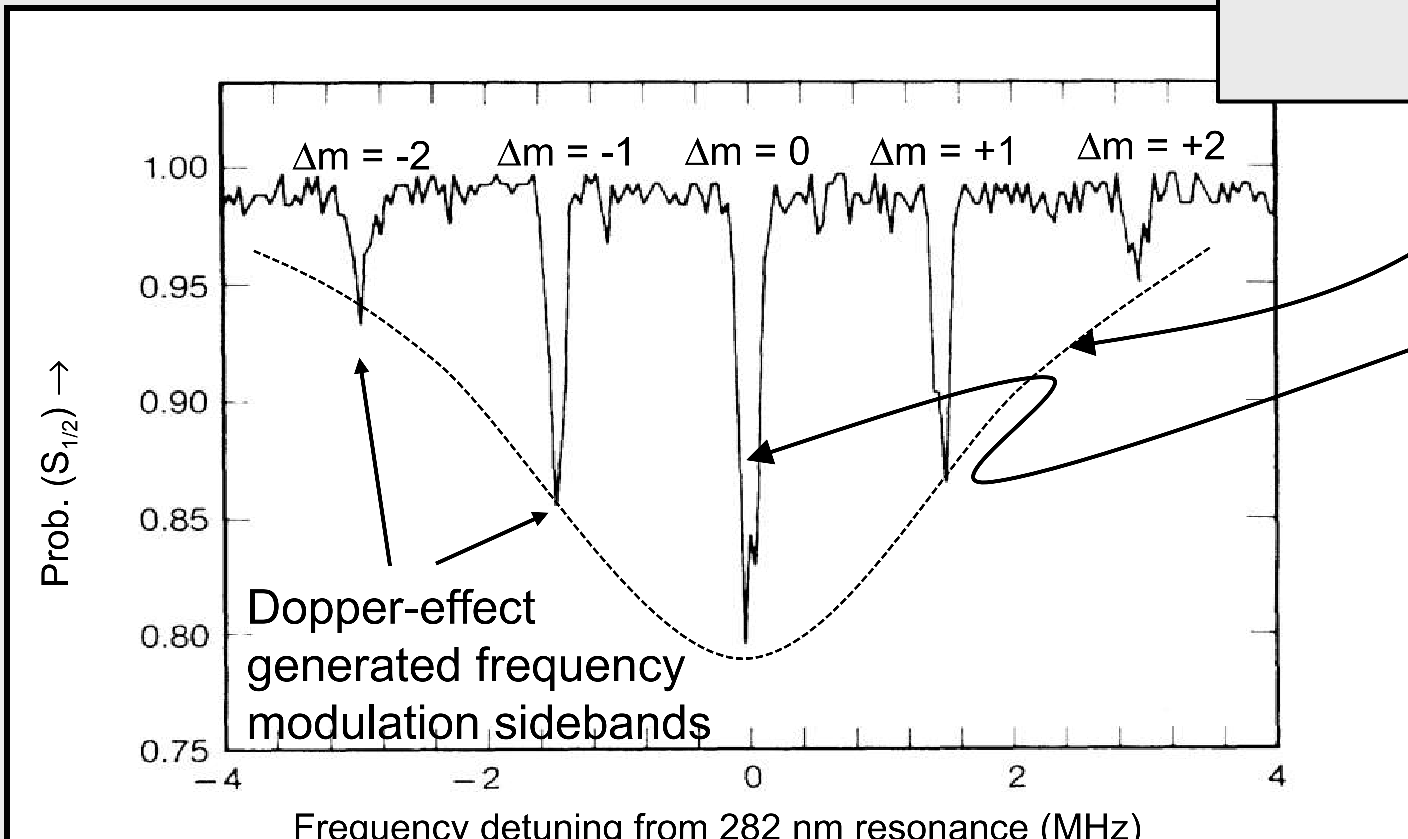
# Optical Mössbauer effect:

for (Doppler) cooling and detection




Doppler profile (T ~ mK)

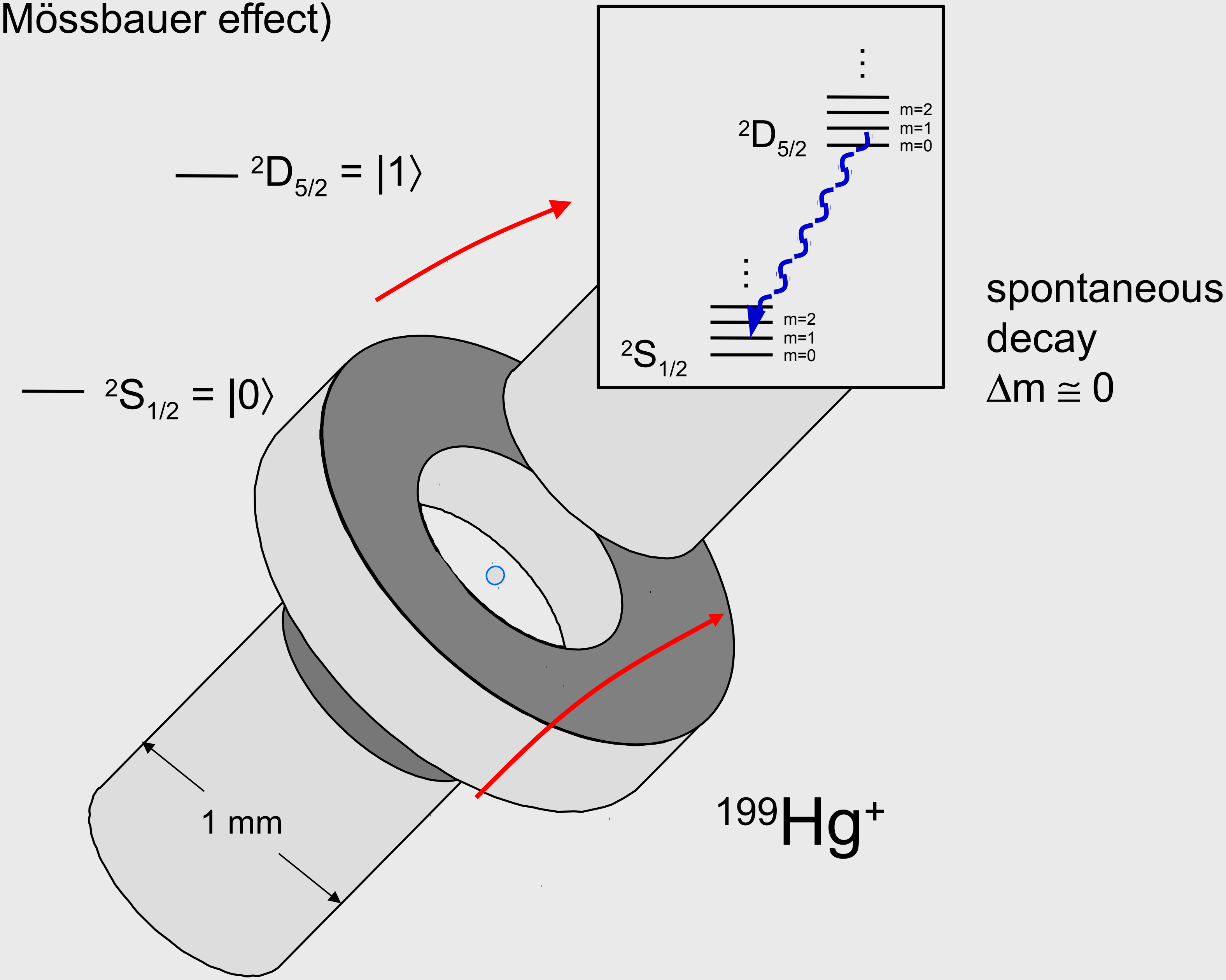
"recoilless" absorption

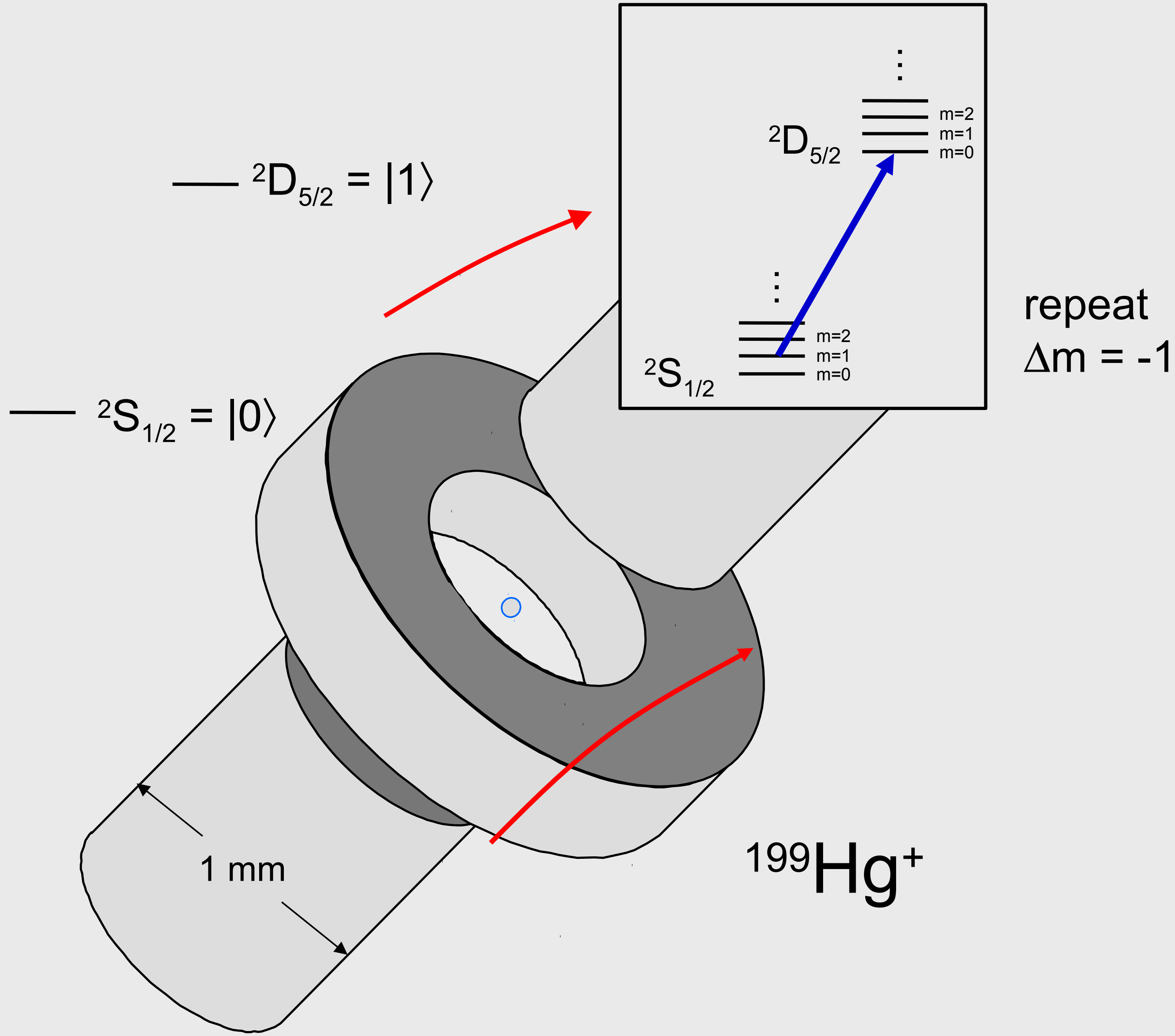



J. C. Bergquist, W. M. Itano, D. J. Wineland, Phys. Rev. A**36**, 428 (1987).

# recoilless decay:

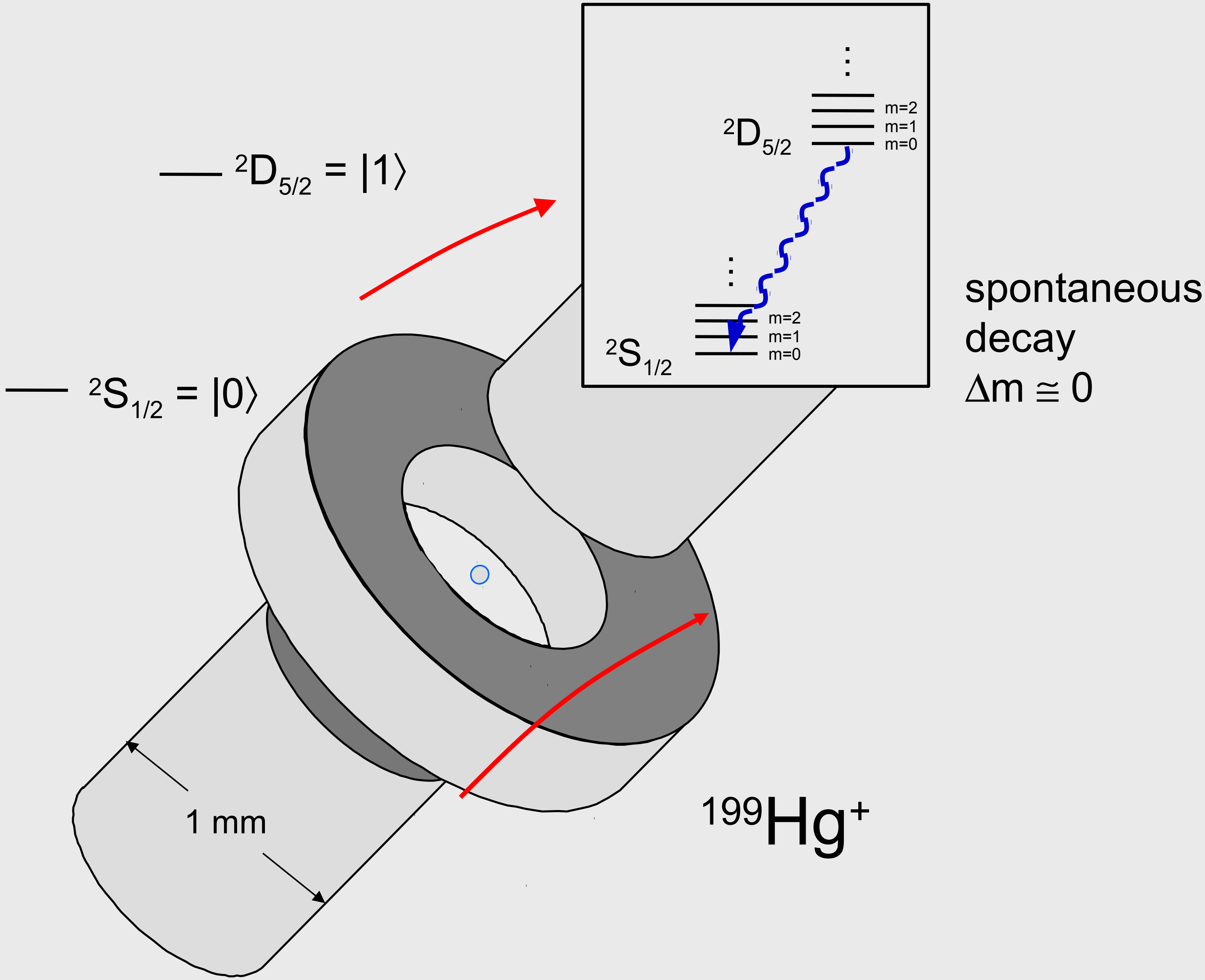
(optical Mössbauer effect)



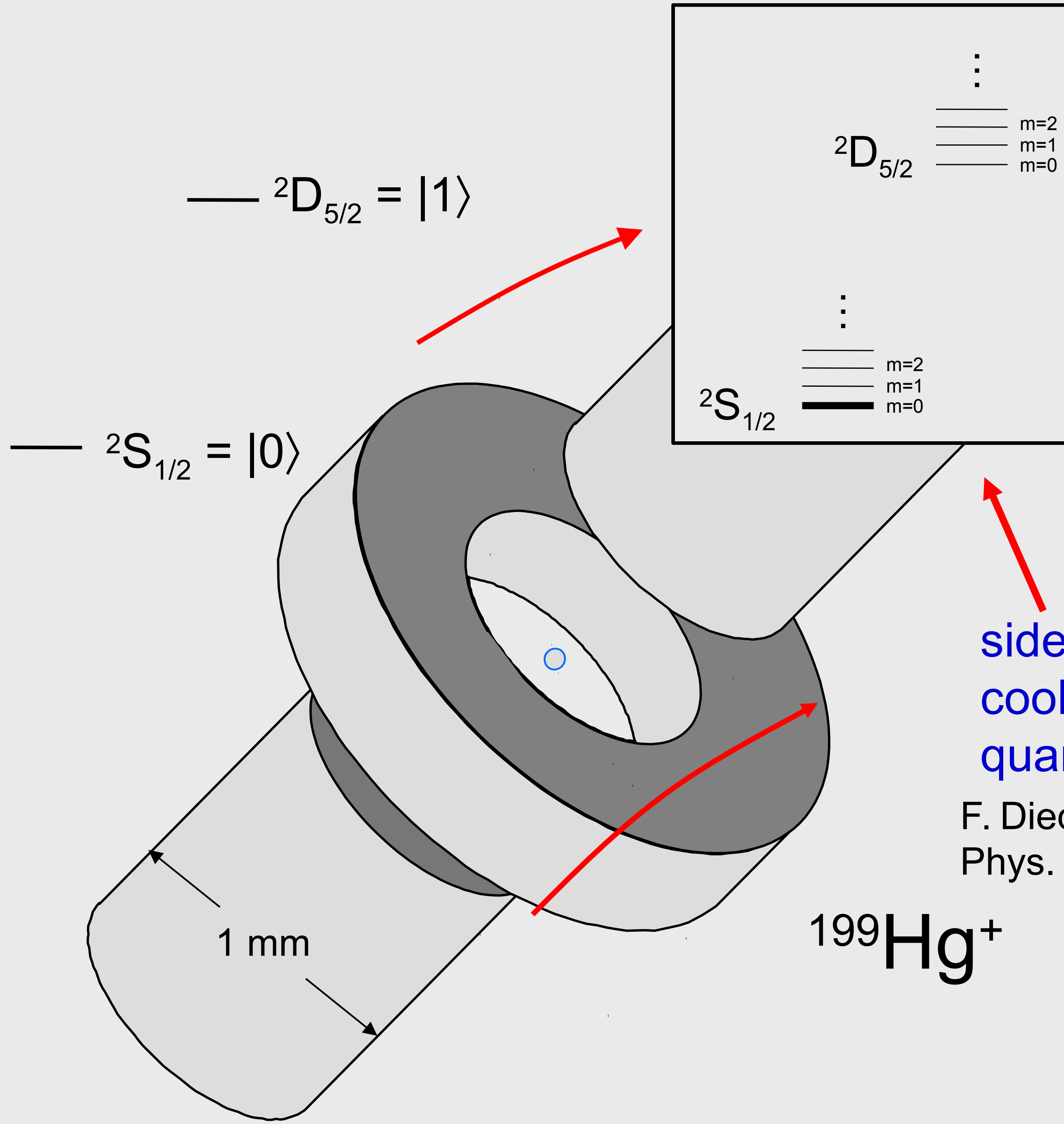


$^2D_{5/2} = |1\rangle$
$^2S_{1/2} = |0\rangle$
$^2D_{5/2}$
m=2
m=1
m=0
$^2S_{1/2}$
m=2
m=1
m=0
repeat
$\Delta m = -1$
1 mm
$^{199}Hg^+$

# fine-scale energy structure:



# fine-scale energy structure:



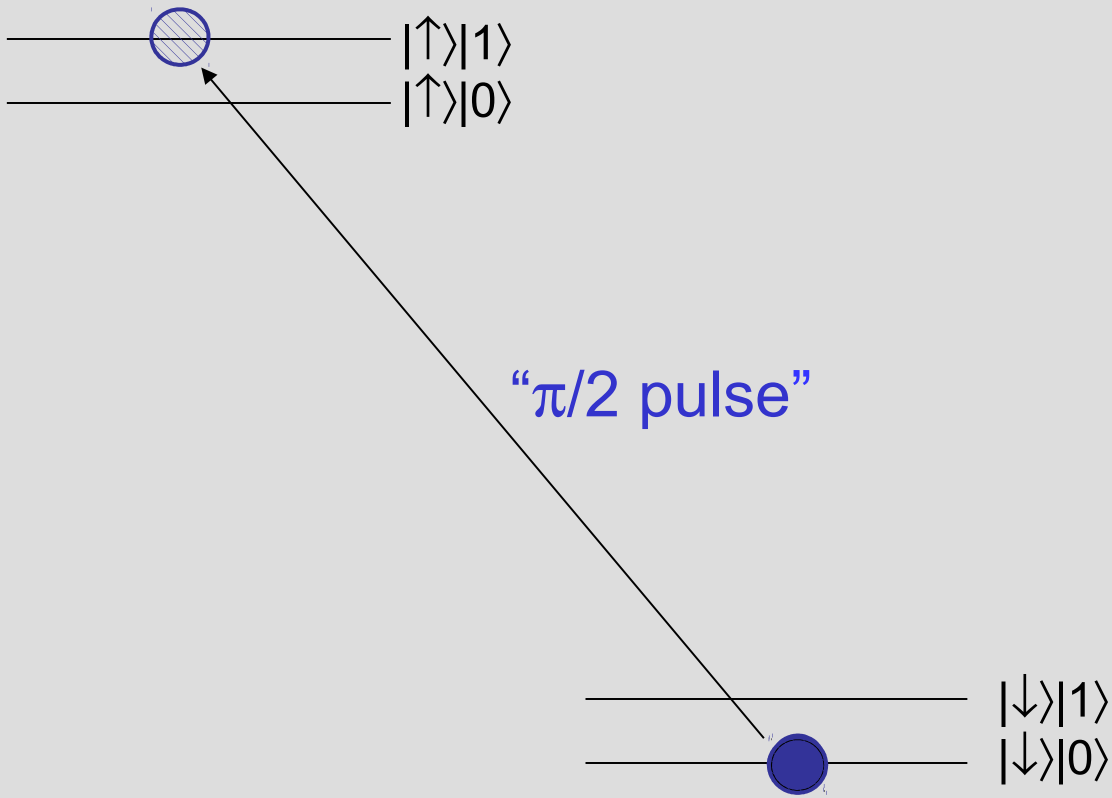

sideband laser cooling:
cool motion to
quantum ground state

F. Diedrich et al.,
Phys. Rev. Lett. 62, 403 (1989)

- Entanglement

$$|\downarrow\rangle|0\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|\downarrow\rangle|0\rangle + |\uparrow\rangle|1\rangle\right]$$

$$\Psi \neq \psi_{spin} \otimes \psi_{motion}$$



# Appendix 1:
## single- optical photon transitions, one mode of motion

$$H = H_0 + H_I$$

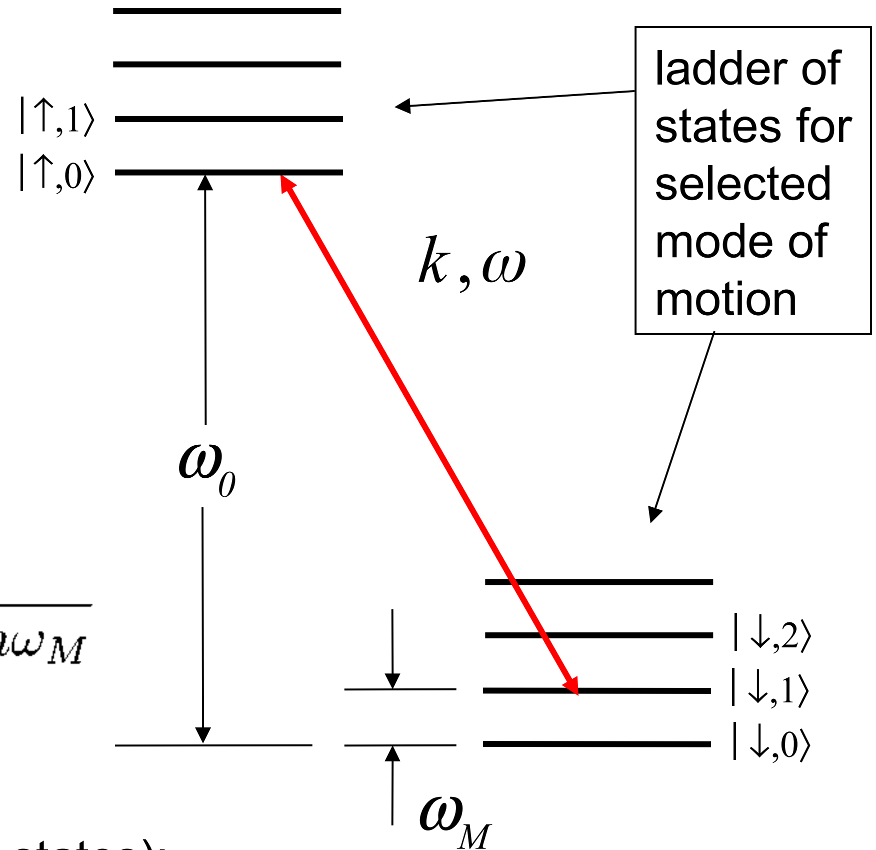
$$H_0 = \hbar\omega_0|\uparrow\rangle\langle\uparrow| + \hbar\omega_M a^\dagger a$$

$$H_I = -e\vec{r}\cdot\vec{E} \quad \text{(electric dipole coupling)}$$

$$\vec{E} = \hat{\epsilon}E_0\cos(\vec{k}\cdot\vec{X}_M - \omega t + \phi)$$

$$\vec{X}_M = \vec{X}_{M0} + \hat{x}x_0(a + a^\dagger), \quad x_0 = \sqrt{\hbar/2m\omega_M}$$

zero-point wavefunction spread




In interaction picture (for internal and motional states):

$$H_I' = \hbar\Omega S_+ \exp(i[\eta(ae^{-i\omega_M t} + a^\dagger e^{i\omega_M t}) - \delta t + \phi + \phi_s]) + h.c. \qquad \delta \equiv \omega - \omega_0$$

$$\Omega \equiv -\frac{eE_0}{2\hbar}\langle\downarrow|\hat{\epsilon}\cdot\vec{r}|\uparrow\rangle \quad \eta \equiv \vec{k}\cdot\hat{x}x_0 \quad \text{(Lamb – Dicke parameter)}$$

$$\phi_s \equiv \vec{k}\cdot\vec{X}_{M0} \quad \text{(spatial phase factor)} \quad S_+ \equiv |\uparrow\rangle\langle\downarrow| \quad h.c. \equiv \text{Hermitian conjugate}$$


see, for example, D.J.W. *et al.,* J. Res. Natl. Inst. Stand. Technol. **103** (3), 259-328 (1998). (available at www.nist.gov/jres)
& D.J.W *et al. Phil. Trans. R. Soc. Lond. A**361**, 1349-1361 (2003).*

$$H_I' = \hbar\Omega S_+ \exp(i[\eta(ae^{-i\omega_M t} + a^\dagger e^{i\omega_M t})$$

$$-\delta t + \phi + \phi_s]) + h.c.$$

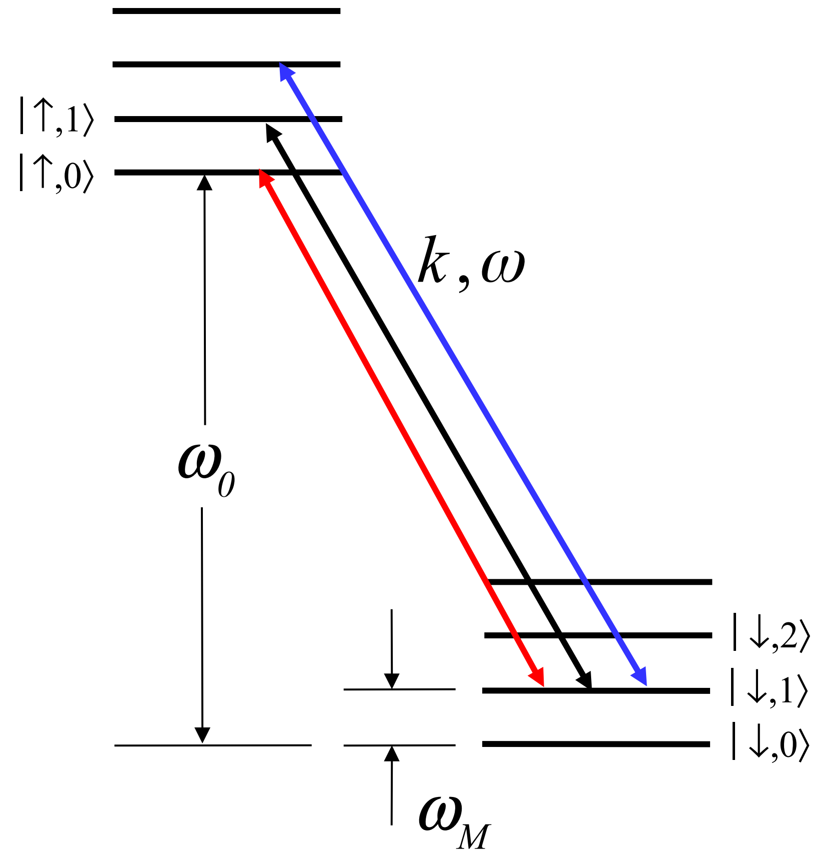
from previous viewgraph

$$\Omega \equiv -\frac{eE_0}{2\hbar}\langle\downarrow|\hat{\epsilon}\cdot\vec{r}|\uparrow\rangle$$

$$\eta \equiv \vec{k}\cdot\hat{x}x_0 \quad \text{(Lamb – Dicke parameter)}$$

$$S_+ \equiv |\uparrow\rangle\langle\downarrow| \qquad \delta \equiv \omega - \omega_0$$

$$\phi_s \equiv \vec{k}\cdot\vec{X}_{M0} \quad \text{(spatial phase factor)}$$




for $\omega_M/2\pi$ = 5 MHz, $x_0 \cong$ 10 nm
for $\lambda$ = 313 nm, $\eta = kx_0 = 2\pi x_0/\lambda \cong 0.20$

expand $H'_I$ in powers of $\eta$. To first order in $\eta$ :

$$H_I' \simeq \hbar\Omega S_+ e^{i(\phi+\phi_s)}\left[e^{-i\delta t} + i\eta a e^{-i(\delta+\omega_M)t} + i\eta a^\dagger e^{-i(\delta-\omega_M)t}\right] + h.c.$$

look for resonant terms:

$\delta = 0$ (“carrier”): $H_I' \simeq \hbar\Omega\left[S_+ e^{i(\phi+\phi_s)} + S_- e^{-i(\phi+\phi_s)}\right]$

$\delta = -\omega_M$ (“red sideband”): $H_I' \simeq \hbar\eta\Omega\left[S_+ a e^{i(\phi+\phi_s+\pi/2)} + S_- a^\dagger e^{-i(\phi+\phi_s+\pi/2)}\right]$

$\delta = +\omega_M$ (“blue sideband”): $H_I' \simeq \hbar\eta\Omega\left[S_+ a^\dagger e^{i(\phi+\phi_s+\pi/2)} + S_- a e^{-i(\phi+\phi_s+\pi/2)}\right]$

$$H_I' = \hbar\Omega S_+ exp(i[\eta(ae^{-i\omega_M t} + a^\dagger e^{i\omega_M t})$$

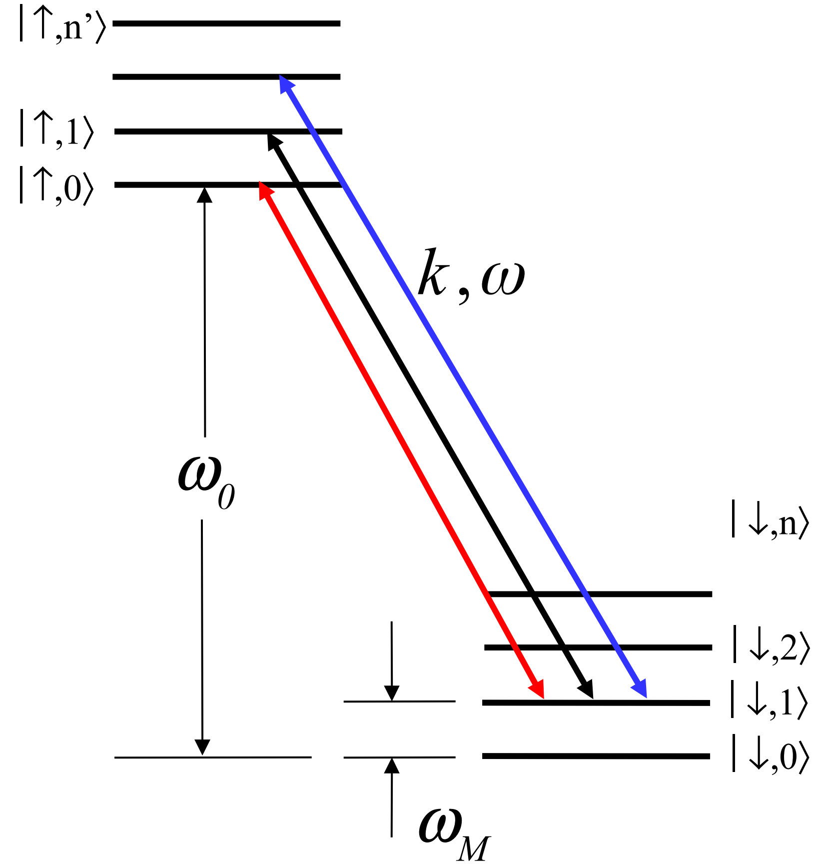
$$-\delta t + \phi + \phi_s]) + h.c.$$

$$\Omega \equiv -\frac{eE_0}{2\hbar}\langle\downarrow|\hat{\epsilon}\cdot\vec{r}|\uparrow\rangle$$

$$\eta \equiv \vec{k}\cdot\hat{x}x_0 \quad (\text{Lamb} - \text{Dicke parameter})$$

$$S_+ \equiv |\uparrow\rangle\langle\downarrow| \qquad \delta \equiv \omega - \omega_0$$

$$\phi_s \equiv \vec{k}\cdot\vec{X}_{M0} \quad (\text{spatial phase factor})$$




for all cases:

$$|\downarrow, n\rangle \rightarrow \cos\Omega_{n,n'}t|\downarrow, n\rangle + e^{i(\phi+\phi_s+\frac{\pi}{2}(|n-n'|-1))}\sin\Omega_{n,n'}t|\uparrow, n'\rangle$$

$$|\uparrow, n'\rangle \rightarrow \cos\Omega_{n,n'}t|\uparrow, n'\rangle + e^{-i(\phi+\phi_s+\frac{\pi}{2}(|n-n'|+1))}\sin\Omega_{n,n'}t|\downarrow, n\rangle$$

"Rabi flopping," qubit rotations

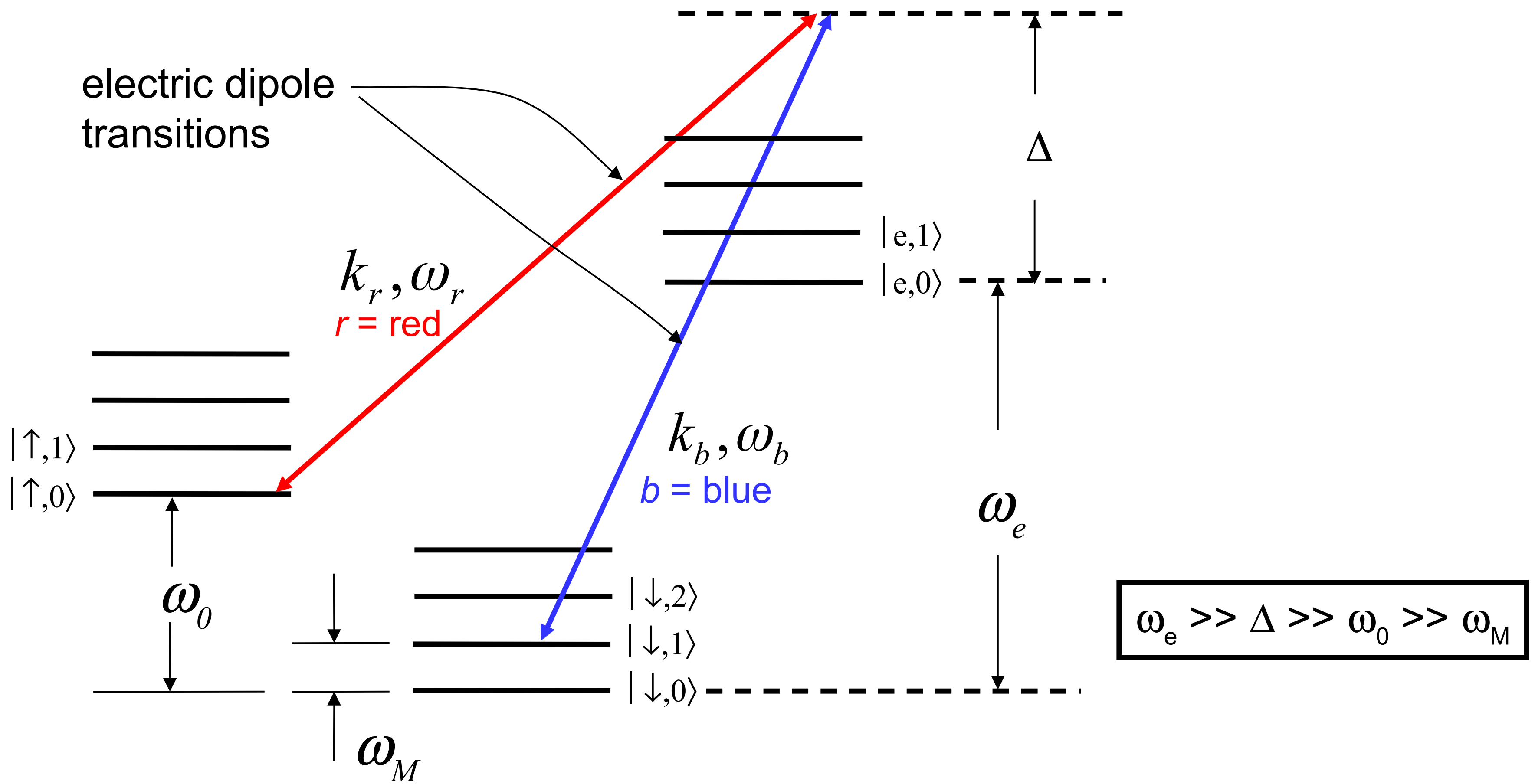
$$\Omega_{n,n'} = \Omega_{n',n} = \Omega|\langle n'|e^{i\eta(a+a^\dagger)}|n\rangle| = \Omega e^{-\frac{\eta^2}{2}}\sqrt{\frac{n_<!}{n_>!}}\eta^{|n-n'|}L_{n_<}^{|n-n'|}(\eta^2)$$

$n_<$ ($n_>$) = smaller (larger) of $n$, $n'$

$$\text{for } \eta \ll 1,\ \Omega_{n,n+1} = \eta\Omega\sqrt{n+1}\ (\text{blue s.b.}),\ \Omega_{n,n-1} = \eta\Omega\sqrt{n},\ (\text{red s.b.})$$

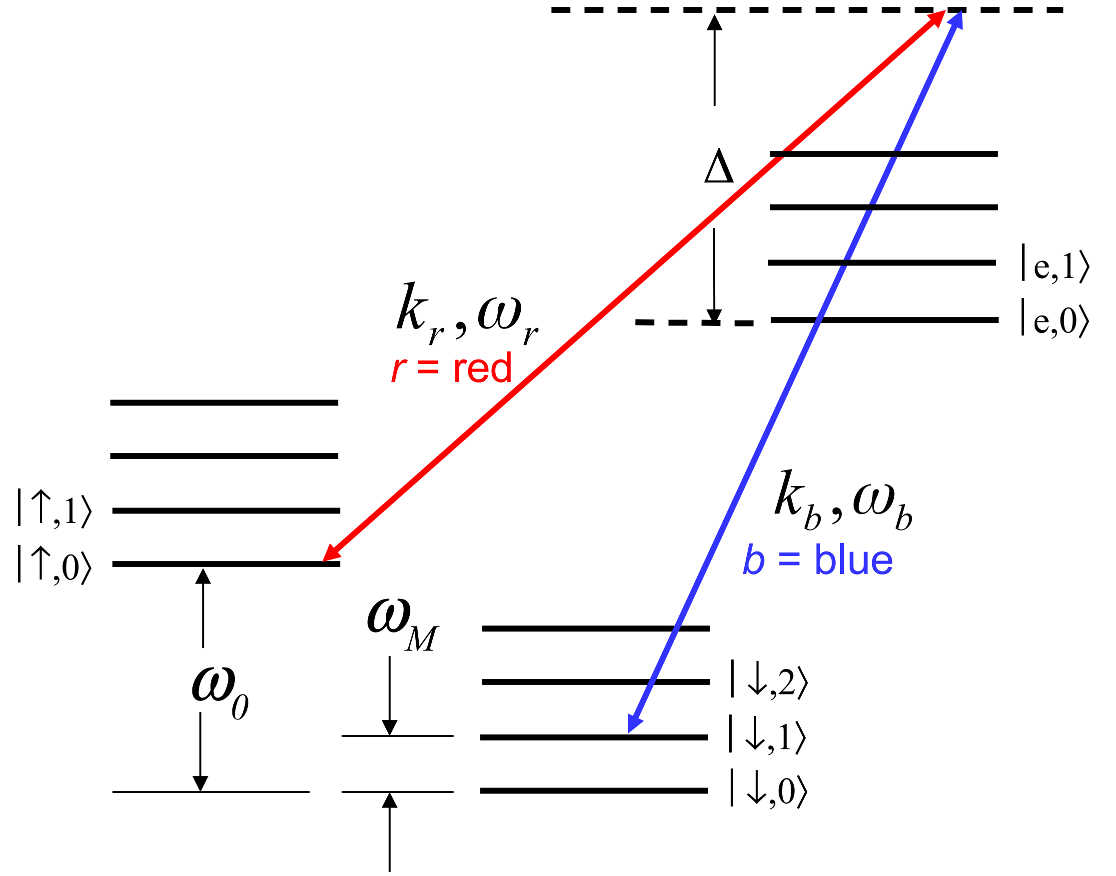
# stimulated-Raman transitions, one mode of motion

e.g., hyperfine qubits




In Be$^+$, $\omega_e/2\pi \simeq 10^{15}$ Hz, $\Delta/2\pi \simeq 10^{11}$ Hz, $\omega_0/2\pi \simeq 1.25$ GHz, $\omega_M/2\pi \simeq 5$ MHz

For qubit rotations, can make $\vec{k}_b$ parallel to $\vec{k}_r$. Spatial phase corresponds to RF wavelength.




In Lamb-Dicke limit, evolution given by:

$$H_I' \simeq \hbar\Omega S_+ e^{i(\phi+\phi_s)}\left[e^{-i\delta t} + i\eta a e^{-i(\delta+\omega_M)t} + i\eta a^\dagger e^{-i(\delta-\omega_M)t}\right] + h.c.$$

(see Appendix 1)

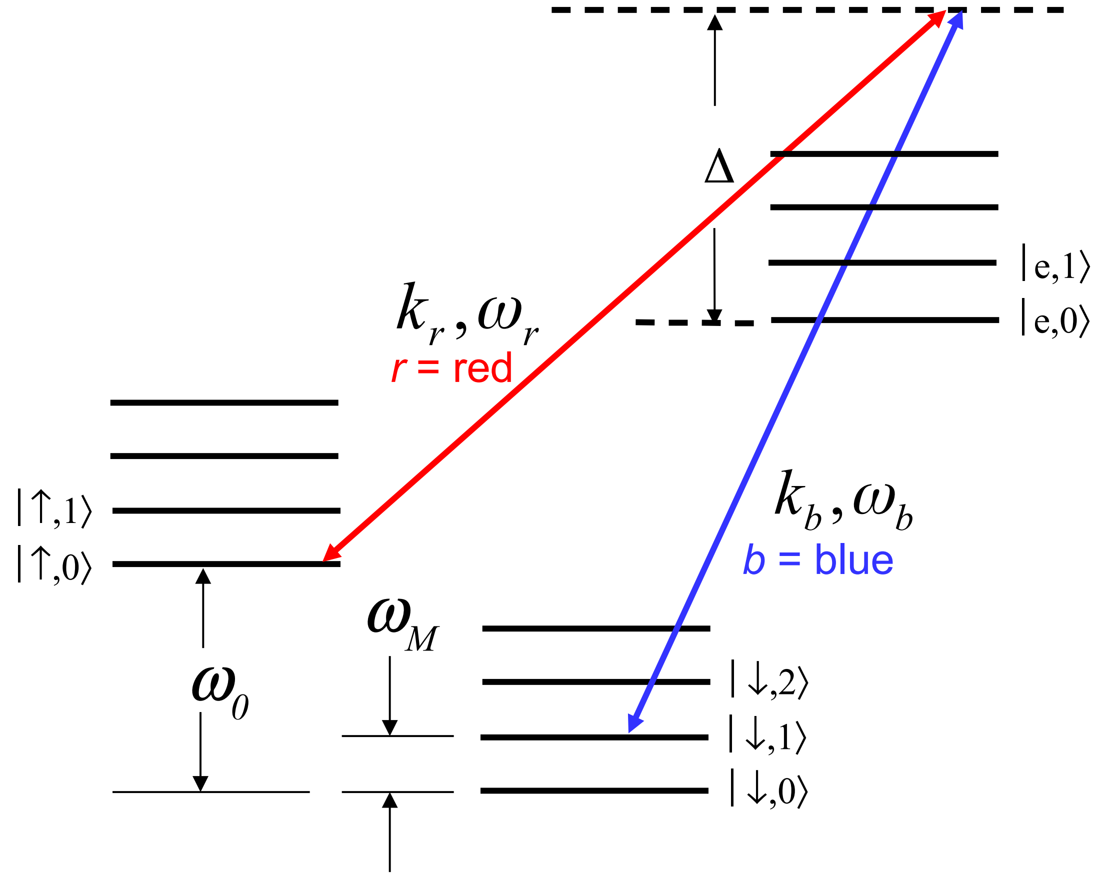
$$\phi_s \equiv (\vec{k}_b - \vec{k}_r)\cdot\vec{X}_{M0} \quad \text{(spatial phase factor)}$$

$$\eta \equiv (\vec{k}_b - \vec{k}_r)\cdot\hat{x}x_0 \quad \text{(Lamb – Dicke parameter)}$$

$$\phi = \phi_r - \phi_b$$

$$\delta \equiv (\omega_b - \omega_r) - \omega_0$$

control phase and frequency of modulator




$$\Omega_{n,n'} = \Omega_{n',n} \equiv \Omega\langle n|e^{-i(\vec{k}_b-\vec{k}_r)\cdot\vec{x}}|n'\rangle = \Omega\langle n|e^{-i\eta(a+a^\dagger)}|n'\rangle$$

$$\Omega \equiv g_b g_r^*/\Delta \qquad g_b \equiv -\langle\downarrow|\hat{\epsilon}_b\cdot\vec{r}\,|e\rangle\frac{eE_{b0}}{2\hbar} \qquad g_r \equiv -\langle\downarrow|\hat{\epsilon}_r\cdot\vec{r}\,|e\rangle\frac{eE_{r0}}{2\hbar}$$

recall, for single photon transitions: $\Omega \equiv -\frac{eE_0}{2\hbar}\langle\downarrow|\hat{\epsilon}\cdot\vec{r}|\uparrow\rangle$

e.g., $Be^+$: P = 1 mW, $w_0$ = 25 μm, $\Delta/2\pi$ = 100 GHz, $\Omega/2\pi$ ~ 0.5 MHz

$$\Omega_{n,n'} \equiv \Omega\langle n|e^{-i(\vec{k}_b-\vec{k}_r)\cdot\vec{x}}|n'\rangle = \Omega\langle n|e^{-i\eta(a+a^\dagger)}|n'\rangle = \Omega_{n',n}$$

$$\Omega \equiv g_b g_r^*/\Delta \qquad \eta \equiv (\vec{k}_b - \vec{k}_r)\cdot\hat{x}x_0$$ (Lamb-Dicke parameter)

$$\langle n|e^{-i\eta(a+a^\dagger)}|n'\rangle = e^{-\eta^2/2}\sqrt{\frac{n_<!}{n_>!}}[-i\eta]^{|n'-n|}L_{n_<}^{|n'-n|}(\eta^2)$$

[$n_<$ ($n_>$) = smaller (larger) of $n$, $n'$]

$$\simeq \langle n|1 - i\eta(a+a^\dagger) - \frac{\eta^2}{2}(1+2\tilde{n}+a^2+(a^\dagger)^2)|n'\rangle \quad \text{to second order in } \eta$$

Carrier transitions:

$$\Omega_{n,n} \simeq \Omega(1-\eta^2(n+1/2)) \simeq \Omega \underbrace{e^{-\eta^2/2}}(1-n\eta^2)$$

Debye-Waller factor – suppression of Rabi frequency from motion averaging over laser wave

Sideband transitions: n' = n ± 1

$$\Omega_{n',n} \simeq -i\Omega\eta\sqrt{n_>}$$

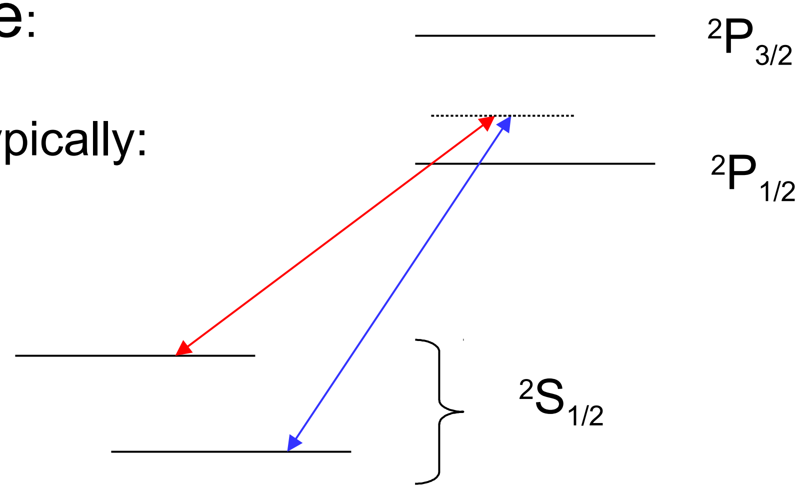
red sideband (*n'* = *n*-1): can get from $H_I = \hbar\eta\Omega(|\downarrow\rangle\langle\uparrow|a^\dagger + h.c.$

Jaynes-Cummings Hamiltonian from cavity-QED

(see, e.g., Raimond, Brune, Haroche, Rev. Mod. Phys. **73**, 565 ('01))

# More complete picture:

- Sum over excited states, typically:




  - must account for differential Stark shifts
  - must account for polarization sensitivity

- For N ions, consider effects of 3N modes

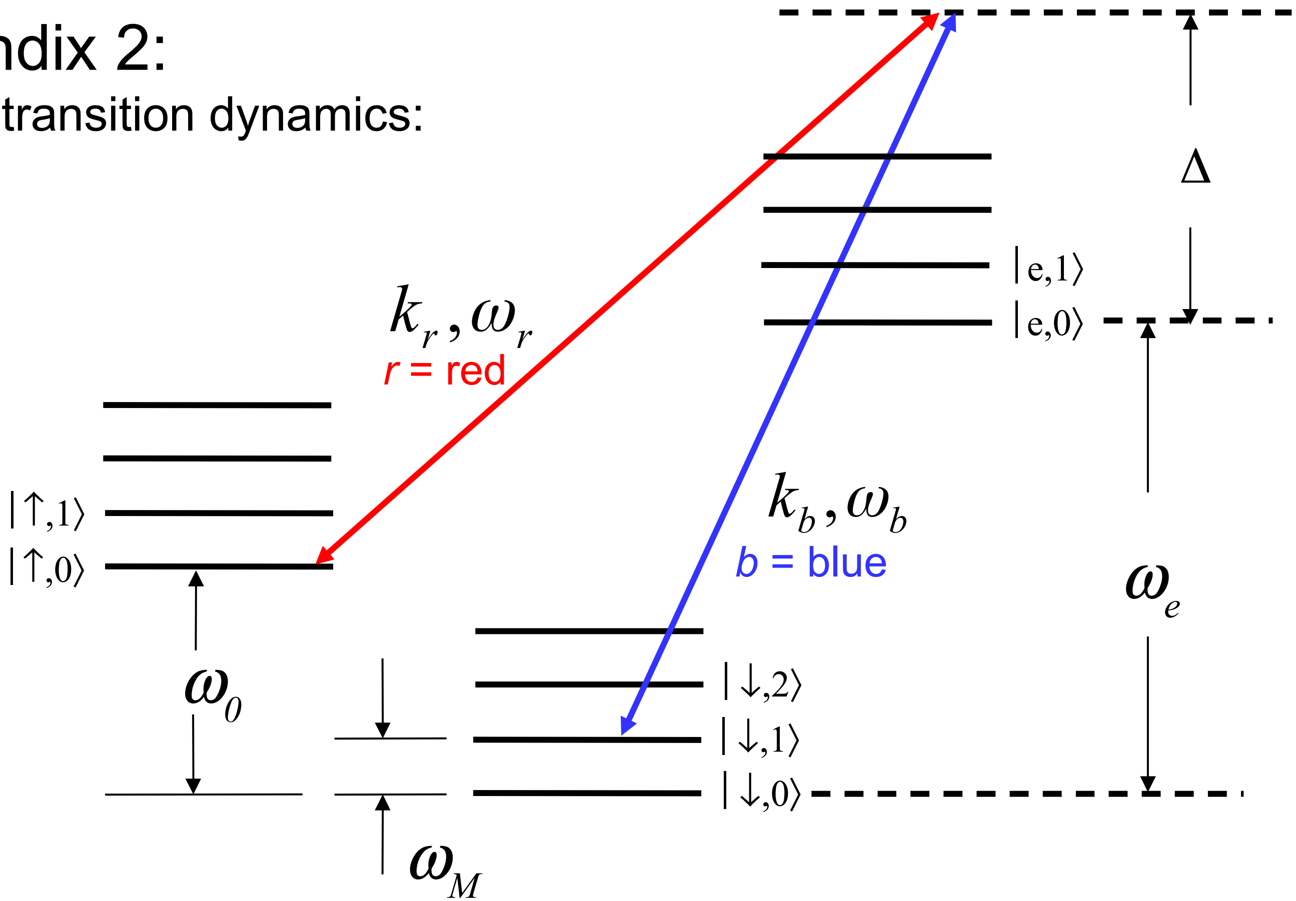
  - account for Debye-Waller factors from "spectator" modes

$$\Omega_{n,n'} = \Omega\langle n|e^{-i\eta(a+a^\dagger)}|n'\rangle \rightarrow \Omega_{n'_k,n_k}\langle\{n_{p\neq k}\}|\prod_{p\neq k} e^{-i\eta_p(a_p+a_p^\dagger)}|\{n_{p\neq k}\}\rangle$$

  - sideband transitions: account for interference from two-mode transitions: e.g. $n\omega_p - m\omega_r \approx \omega_M$, (n, m integers)

# Appendix 2:

Raman-transition dynamics:




$$H = H_0 + H_I$$

$$H_0 = \hbar\omega_0|\uparrow\rangle\langle\uparrow| + \hbar\omega_e|e\rangle\langle e| + \hbar\omega_M a^\dagger a \qquad H_I = -\sum_{i=b,r} e\vec{r}\cdot\vec{E}_i$$

$$\vec{E}_i = \hat{\epsilon}_i E_{i0}\cos(\vec{k}_i\cdot\vec{X}_M - \omega_i t + \phi_i), \qquad i \in \{b, r\}$$

(neglecting $\phi_s$) $\vec{X}_M = \hat{x}x_0(a + a^\dagger), \quad x_0 = \sqrt{\hbar/2m\omega_M}$ ← zero-point wavefunction spread

$$\Psi = \sum_{n=0}^{\infty} \left[ C_{\downarrow,n} e^{-in\omega_M t} |\downarrow, n\rangle + C_{\uparrow,n} e^{-i[\omega_0 + n\omega_M]t} |\uparrow, n\rangle + C_{e,n} e^{-i[\omega_e + n\omega_M]t} |e, n\rangle \right]$$

$\langle \downarrow, n | (i\hbar \partial\Psi/\partial t = H\Psi)$ (+ rotating wave approximation) ⇒

$$\dot{C}_{\downarrow,n} = i g_b \sum_{m=0}^{\infty} e^{i(\omega_b - \omega_e + (n-m)\omega_M)t} \langle n | e^{-i\vec{k}_b \cdot \vec{X}} | m \rangle C_{e,m}$$

similarly:

$$\dot{C}_{\uparrow,n} = i g_r \sum_{m=0}^{\infty} e^{i(\omega_r - (\omega_e - \omega_0) + (n-m)\omega_M)t} \langle n | e^{-i\vec{k}_r \cdot \vec{X}} | m \rangle C_{e,m}$$

$$\dot{C}_{e,m} = i g_b^* \sum_{p=0}^{\infty} e^{-i(\omega_b - \omega_e - (m-p)\omega_M)t} \langle m | e^{i\vec{k}_b \cdot \vec{X}} | p \rangle C_{\downarrow,p}$$

$$+ i g_r^* \sum_{p=0}^{\infty} e^{-i(\omega_r - (\omega_e - \omega_0) - (m-p)\omega_M)t} \langle m | e^{i\vec{k}_r \cdot \vec{X}} | p \rangle C_{\uparrow,p}$$

$$g_b \equiv \langle \downarrow | \hat{\epsilon}_b \cdot \vec{r} \, | e \rangle \frac{e E_{b0} e^{-i\phi_b}}{2\hbar} \qquad g_r \equiv \langle \downarrow | \hat{\epsilon}_r \cdot \vec{r} \, | e \rangle \frac{e E_{r0} e^{-i\phi_r}}{2\hbar}$$

$$\dot{C}_{e,m} = ig_b^* \sum_{p=0}^{\infty} e^{-i(\omega_b - \omega_e - (m-p)\omega_M)t} \langle m | e^{i\vec{k}_b \cdot \vec{X}} | p \rangle C_{\downarrow,p}$$

$$+ig_r^* \sum_{p=0}^{\infty} e^{-i(\omega_r - (\omega_e - \omega_0) - (m-p)\omega_M)t} \langle m | e^{i\vec{k}_r \cdot \vec{X}} | p \rangle C_{\uparrow,p}$$

"Adiabatic elimination":

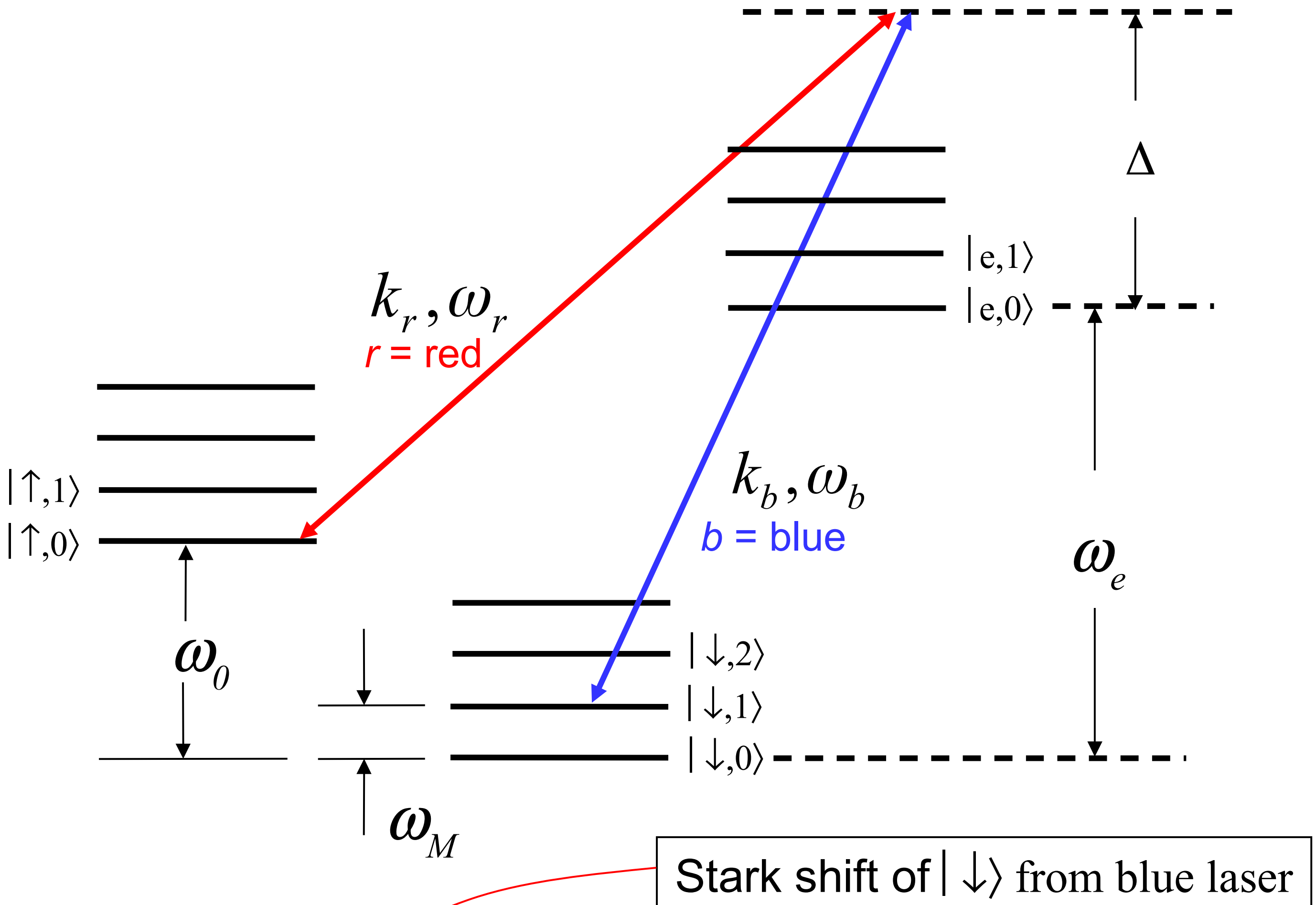
$$C_{e,m} \equiv e^{-i\Delta t} C'_{e,m}; \quad \dot{C}_{e,m} = e^{-i\Delta t}(\dot{C'}_{e,m} - i\Delta C'_{e,m})$$

make ansatz: $\Delta C'_{e,m} \gg \dot{C'}_{e,m}$ (can check later)

$$\Rightarrow C'_{e,m} = ie^{i\Delta t}\dot{C}_{e,m}/\Delta \qquad \Rightarrow C_{e,m} = i\dot{C}_{e,m}/\Delta$$

$$\dot{C}_{\downarrow,n} = ig_b \sum_{m=0}^{\infty} e^{i(\omega_b - \omega_e + (n-m)\omega_M)t} \langle n | e^{-i\vec{k}_b \cdot \vec{X}} | m \rangle C_{e,m}$$

$$\dot{C}_{\uparrow,n} = ig_r \sum_{m=0}^{\infty} e^{i(\omega_r - (\omega_e - \omega_0) + (n-m)\omega_M)t} \langle n | e^{-i\vec{k}_r \cdot \vec{X}} | m \rangle C_{e,m}$$




$$\dot{C}_{\downarrow,n} = -i\frac{|g_b|^2}{\Delta}C_{\downarrow,n} - i\sum_{n'=0}^{\infty}\Omega_{n,n'}\; e^{i(\delta-(n'-n)\omega_M)t}C_{\uparrow,n'}$$

$$\delta \equiv \omega_b - \omega_r - \omega_0$$

$$\dot{C}_{\uparrow,n'} = -i\frac{|g_r|^2}{\Delta}C_{\uparrow,n'} - i\sum_{n=0}^{\infty}\Omega^*_{n',n}\; e^{-i(\delta+(n-n')\omega_M)t}C_{\downarrow,n}$$

Add in other Stark shifts

$$\dot{C}_{\downarrow,n} = -i\Delta_{S\downarrow}C_{\downarrow,n} - i\sum_{p=0}^{\infty}\Omega_{n,p}\ e^{i(\delta-(p-n)\omega_M)t}C_{\uparrow,p}$$

$$\dot{C}_{\uparrow,n} = -i\Delta_{S\uparrow}C_{\uparrow,n} - i\sum_{p=0}^{\infty}\Omega^*_{n,p}\ e^{-i(\delta+(p-n)\omega_M)t}C_{\downarrow,p}$$

$$\Delta_{S\downarrow} = |g_b|^2/\Delta + |g_{\downarrow,e,r}|^2/(\Delta-\omega_0), \quad \Delta_{S\uparrow} = |g_{\uparrow,e,b}|^2/(\Delta+\omega_0) + |g_r|^2/\Delta$$

absorb Stark shifts into wave function amplitudes

$$C_{\downarrow,n} = C'_{\downarrow,n}e^{-i\Delta\omega_{S\downarrow}t}, \quad C_{\uparrow,n} = C'_{\uparrow,n}e^{-i\Delta\omega_{S\uparrow}t}$$

near a resonance:

$$\delta_{n',n} \equiv \delta - (\Delta\omega_{S\uparrow} - \Delta\omega_{S\downarrow}) - (n'-n)\omega_M \ \simeq 0 \quad (\delta \equiv \omega_b - \omega_r - \omega_0)$$

$$\dot{C}'_{\downarrow,n} = -i\Omega_{n,n'}e^{i\delta_{n',n}t}C'_{\uparrow,n'} \quad \dot{C}'_{\uparrow,n'} = -i\Omega^*_{n',n}e^{-i\delta_{n',n}t}C'_{\downarrow,n}$$

⇒ Rabi flopping (for $\delta = 0$, exact resonance)

$$\ddot{C}'_{\downarrow,n} + |\Omega_{n',n}|^2C'_{\downarrow,n} = 0, \quad \ddot{C}'_{\uparrow,n'} + |\Omega_{n',n}|^2C'_{\uparrow,n'} = 0$$

(except for Stark shifts) evolution given (in Lamb-Dicke limit) by Hamiltonian

$$H_I' \simeq \hbar\Omega S_+ e^{i(\phi+\phi_s)}\left[e^{-i\delta t} + i\eta a e^{-i(\delta+\omega_M)t} + i\eta a^\dagger e^{-i(\delta-\omega_M)t}\right] + h.c.$$

$$\delta \equiv (\omega_b - \omega_r) - \omega_0 \quad \phi = \phi_r - \phi_b$$

with:

$$\phi_s \equiv (\vec{k}_b - \vec{k}_r)\cdot \vec{X}_{M0} \quad \text{(spatial phase factor)}$$

$$\eta \equiv (\vec{k}_b - \vec{k}_r)\cdot \hat{x} x_0 \quad \text{(Lamb – Dicke parameter)}$$

$$\Omega_{n,n'} \equiv \Omega\langle n|e^{-i(\vec{k}_b-\vec{k}_r)\cdot\vec{X}}|n'\rangle = \Omega\langle n|e^{-i\eta(a+a^\dagger)}|n'\rangle = \Omega_{n',n}$$

$$\Omega \equiv g_b g_r^*/\Delta \qquad g_b \equiv -\langle\downarrow|\hat{\epsilon}_b\cdot\vec{r}\,|e\rangle\frac{eE_{b0}}{2\hbar} \qquad g_r \equiv -\langle\downarrow|\hat{\epsilon}_r\cdot\vec{r}\,|e\rangle\frac{eE_{r0}}{2\hbar}$$

recall, for single photon transitions: $\Omega \equiv -\frac{eE_0}{2\hbar}\langle\downarrow|\hat{\epsilon}\cdot\vec{r}|\uparrow\rangle$